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Resources and environmental systems management under synchronic interval uncertainties

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Abstract Resources and environmental systems management (*RESM*) is challenged by the synchronic effects of interval uncertainties in the related practices. The synchronic interval uncertainties are misrepresented as random variables, fuzzy sets, or interval numbers in conventional *RESM* programming techniques including stochastic programming. This may lead to ineffectiveness of resources allocation, high costs of recourse measures, increased risks of unreasonable decisions, and decreased optimality of system profits. To fill the gap of few corresponding studies, a synchronic interval linear programming (*SILP*) method is proposed in this study. The proposition of *interval sets* and *interval functions* and coupling them with linear programming models lead to development of an *SILP* model for *RESM*. This enables incorporation of interval uncertainties in resource constraints and synchronic interval uncertainties in the programming objective into the optimization process. An analysis of the distribution-independent geometric properties of the feasible regions of *SILP* models results in proposition of *constraint violation likelihoods*. The tradeoff between system optimality and constraint violation is analyzed. The overall optimality of *SILP* systems under synchronic intervalness is quantified through proposition of *integrally optimal solutions*. Integration of these efforts leads to a violation-constrained interval integral method for optimization of *RESM* systems under synchronic interval uncertainties. Comparisons with selected existing methods reveal the effectiveness of *SILP* at eliminating negativity of synchronic intervalness, enabling risk management of and achieving overall optimality of *RESM* systems, and enhancing the reliability of optimization techniques for *RESM* problems. The exploited framework for analyzing synchronic interval uncertainties in *RESM* systems is helpful for addressing synchronisms of other uncertainties such as randomness or fuzziness and avoiding the resultant decision mistakes and disasters due to neglecting them.

Keywords Resources and environmental systems management · Synchronic interval uncertainty · Interval linear programming · Interaction · Risk analysis

1 Introduction

Linear programming (*LP*) (Kantorovich 1940) is an effective decision support tool for resources and environmental systems management (*RESM*) (Anderson et al. 2015; Banos et al. 2011; Psacharopoulos 2014; Lin et al. 2012; Ahmad et al. 2014; Lee 2011; Ivanov et al. 2012; Tang and Zhou 2012; Mérel and Howitt 2014). Simplicity of *LP* models, e.g. linearity of parameter relationships, singleness of optimization objectives, continuity of parameter values, independence of constraints, certainty of coefficient estimations and invariability of model features, increases their applicability to *RESM* problems, but also

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decreases reliability of the optimization process due to diverse complexities in *RESM* systems (Peter and Mayer 1976; Jamison and Lodwick 2001; Kahraman 2008; Birge and Louveaux 2011; Quaeghebeur et al. 2012; Luhandjula 2014; Dong et al. 2014a, b, c, Dong et al. 2015). One of the most challenging complexities is synchronic interval uncertainty.

Ineffectiveness of estimation techniques, insufficiency of data support, unpredictability of system noises, or other potential reasons may cause the emergence of interval uncertainty in the process of estimating *LP* model coefficients for future planning of *RESM* systems. It is of high likeliness that fluctuation ranges, of which the distributional information is unknown, are the only reliable estimation of these uncertain coefficients (Dong et al. 2013a, b). Accordingly, a series of interval linear programming (*ILP*) methods (Charnes et al. 1977; Ishibuchi and Tanaka 1990; Huang et al. 1992; Inuiguchi 1993; Tong 1994; Inuiguchi and Sakawa 1995, 1997; Chanas and Kuchta 1996; Sengupta and Pal 2000; Chinneck and Ramadan 2000; Inuiguchi et al. 2003; Dong et al. 2011, 2012; Cheng et al. 2015a, b, 2017) were proposed to reflect the reality that uncertain coefficients may fluctuate within known ranges under unknowns of distributions. Particularly, Huang et al. (1992) proposed the *ILP* model as an integration of an *LP* model and interval coefficients and developed a two-step solution (*TSS*) algorithm relying on two dependent *LP* sub-models to solve the *ILP* model. A problem of *TSS* was that interval-coefficient constraints in the *ILP* model might be violated by the obtained solutions in some cases, leading to decreased reliability of *TSS*-based decision support. Accordingly, Cheng et al. (2015b) exploited an interval recourse linear programming (*IRLP*) approach which was verified to be effective in resolving this problem, reproducing the largest decision space excluding infeasible solutions, and enhancing the reliability of decision support for *RESM*. Due to high feasibility and reliability, *ILP* has been gained extensive attentions of scholars all over the world and has been applied to many *RESM* studies (e.g. Cheng et al. 2009, 2015a, b, 2017; Dong et al. 2011, 2012, 2013a, b, 2014a, b, c, 2015). A review of these studies was conducted in (Cheng et al. 2015b).

Furthermore, interval uncertainties in *RESM* problems may be of synchronic effects because of the intersection of system components. For instance, the coefficients of two decision variables share an identical component of which the uncertain property is only known as a range. This component may result in synchronisms of the two coefficients. A particular example is presented in the second paragraph of Sect. 3 for readers' convenience. As a particular case of interactions of interval uncertainties, the synchronism is easily identified and quantified. *LP* problems with coefficients being of synchronic interval uncertainties are named as synchronic interval linear programming (*SILP*) problems in this study. Reflecting such a complexity and incorporating it into the decision support process is desired for *RESM* practices. Failure in doing these would misestimate ranges of system optimality, disable reasonability of management schemes, sacrifice optimality of programming systems, and threaten interests of stakeholders in *RESM* systems.

However, synchronic interval uncertainties invalidated existing methods in optimization of *SILP* systems. Previously, few methods were specifically proposed for supporting *SILP* under synchronic interval uncertainties; instead, decision making was mainly based on unreasonable simplifications when employing existing methods. Representative alternative ones consisted of *LP* (Kantorovich 1940; Dantzig 1963), Monte Carlo Analysis (*MCA*) (Metropolis and Ulam 1949; Klaus and Albert 1995; Grinstead and Snell 1997), robust optimization (*RO*) (Ben-Tal and Nemirovski 2002; Bertsimas and Sim 2004; Ben-Tal et al. 2009; Bertsimas et al. 2011; Gabrel et al. 2014), interval linear programming (*ILP*) (Charnes et al. 1977; Ishibuchi and Tanaka 1990; Huang et al. 1992; Inuiguchi 1993; Tong 1994; Inuiguchi and Sakawa 1995, 1997; Chanas and Kuchta 1996; Sengupta and Pal 2000; Chinneck and Ramadan 2000; Inuiguchi et al. 2003; Dong et al. 2011, 2012; Cheng et al. 2015b, 2017), distribution-based linear programming (*DLP*) (Charnes and Cooper 1959; Prékopa 1990; Inuiguchi and Ramík 2000; Ruszczynski and Shapiro 2003) including stochastic linear programming (*SLP*) (Charnes and Cooper 1959; Prékopa 1990; Ruszczynski and Shapiro 2003), and synchronic interval Gaussian mixed-integer programming (*SIGMIP*) (Cheng et al. 2015a).

It is confronted with multiple challenges that these existing methods are applied to solve *SILP* problems. The most critical one is incapability of reflecting the synchronic effect of interval uncertainties in modeling and solving processes, e.g. for *IRLP* (Cheng et al. 2015b) and *SLP* (e.g. Charnes and Cooper 1959; Prékopa 1990; Ruszczynski and Shapiro 2003). Another one is weak effectiveness at addressing interval uncertainties, which varies with techniques. For instance, interval uncertainties are expressed as representative deterministic values in *LP*, but the solutions could hardly be optimal for both the selected values and others. Distributions of uncertain properties, which are originally unknown in *SILP* problems, are constructed based on artificial assumptions in both *MCA* and *DLP* including *SLP*. Imposing nonexistent distributional information to uncertain properties results in the mismatch between real-world problems and human-built models. Resulting solutions are optimal for constructed models, but their optimality could not be guaranteed in practices. Heavy computational loads also restrict the applicability of

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MCA for large-scale *SILP* problems. In *RO*, interval uncertainties are represented as an uncertainty set in which elements are intervals; the solution is too extremely conservative to be acceptable for all decision makers who are diversified in acceptability of constraint violation risks. *ILP* is more effective at reflecting interval uncertainties than other techniques; nevertheless, the tradeoffs between constraint violation and system optimality are hardly evaluated, and the overall optimality of *SILP* systems is unachievable. In comparison with the aforementioned existing methods, *SIGMIP* (Cheng et al. 2015a) that was developed for guiding air quality control under multiple complexities including synchronic intervalness may be more capable of addressing synchronic interval uncertainties in *RESM*; however, *SIGMIP* cannot systematically evaluate the individual impacts of synchronic interval uncertainties for *RESM* practices and be taken as a generalized method for *RESM* under synchronic intervalness; most importantly, *SIGMIP* lacks a solid theoretical basis and a comprehensive analysis of synchronic-interval *RESM* systems, which is not helpful for development of more advanced *SILP* methods.

As a result, execution of the *RESM* schemes obtained through existing system optimization methods (e.g. Cheng et al. 2015a, b) would result in a variety of problems such as ineffective allocation of resources, a decrease of system optimality, violation of constraints, and unexpectedly high costs of recourse measures. Correspondingly, an *SILP* method will be proposed in this study to mitigate the challenge of synchronic interval uncertainties in *RESM* and to fill the gap of few studies specialized in comprehensively addressing this challenge. Specifically:

a. In Sect. 2, the origination, characteristics, influences and quantitative analysis of interval uncertainties in *LP* problems will be reviewed, building a necessary foundation for discussions on *SILP* problems. *Interval sets* will be defined to reflect interval uncertainties. Coupling of *interval sets* and *LP* models will result in the development of interval linear programming (*ILP*) models.
b. In Sect. 3, the origination, influences, and characterization of synchronic interval uncertainties in the programming objective will be analyzed. Based on the definition of *interval function*s, an *SILP* model, equivalent to the integration of an *LP* model, *interval sets,* and *interval function*s, will be proposed to parameterize *RESM* problems under synchronic interval uncertainties.
c. In Sect. 4, an analysis of geometric properties of feasible regions of *SILP* models will lead to proposition of *constraint violation likelihood*s (*CVL*). *CVL* will enable quantification of constraint violation and analysis of risk-profit tradeoffs under intervalness. When the maximum acceptable values of *CVL* are determined by decision makers, constraints of *SILP* models will be converted as linear inequalities or equalities. Equifinality of *CVL*s and constraint violation probabilities will be evaluated to reveal the independence of the effectiveness of *CVL*s at reflecting constraint violation risks.
d. In Sect. 5, the *integrally optimal solution* of an *SILP* model will be proposed to quantify the overall optimality of the model based on multidimensional integral. Equivalence with existing related definitions, further transformation of *SILP* models, and simplification of models under particular cases will be discussed. Integration of efforts in Sects. 4 and 5 will result in exploitation of a violation-constrained interval integral (*VCII*) method for solving *SILP* models. The procedures of *VCII* will be specified in Sect. 5.
e. A simple problem as an abstraction of real-world *RESM* problems will be introduced in Sect. 6 to demonstrate the *SILP* model and the *VCII* solution algorithm. This will facilitate potential users of the developed method, avoid inappropriate usages and unscientific decisions, and eliminate undesired socio-economic or eco-environmental losses in *SILP* practices.
f. In Sect. 7, comparisons between *SILP* and selected existing methods will reveal the effectiveness of *SILP* in multiple aspects, e.g. reflection of interval uncertainties and their synchronic effects, analysis of the tradeoffs between constraint violation risks and system optimality under unknown distributions, identification of the overall optimality of *SILP* systems, and elimination of negative influences of the synchronisms of interval uncertainties. Potential improvements and extensions of *SILP* will also be discussed in this section.

2 Interval uncertainties

Linear programming (*LP*) can be expressed as optimization of a linear objective function subject to linear equalities or inequalities. A generalized formulation is maximization of $f = \boldsymbol{CX}$ subject to $\boldsymbol{AX} \le \boldsymbol{b}$ and $\boldsymbol{X} \ge \boldsymbol{0}$ where $\boldsymbol{C} = \{c_j \mid j = 1, 2, \ldots, n\}_{1\times n}$, $\boldsymbol{A} = \{a_{ij} \mid i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}_{m\times n}$, $\boldsymbol{b} = \{b_i \mid i = 1, 2, \ldots, m\}_{m\times 1}$; $\boldsymbol{C}$, $\boldsymbol{A}$ and $\boldsymbol{b}$ are matrices or vectors of real numbers; and $\boldsymbol{X} = \{x_j \mid j = 1, 2, \ldots, n\}_{n\times 1}$ is a vector of real-valued decision variables x_j $(j = 1, 2, \ldots, n)$. A programming problem that can be represented as an *LP* model is named as an *LP* problem in this study.

The effectiveness of *LP* models is challenged by interval uncertainties of properties of components in many real-world *RESM* problems. For instance, the daily generation rate of solid wastes is required for future planning of a municipal solid waste system. The real value is unknown,

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distributional information is unachievable, and the only reliable estimation is a range, e.g. [100, 120] tonne/day. Interval uncertainty, which is also intervalness in this study, originates from intrinsic inconsistent characteristics and human cognitive limitations. System properties vary in the planning period under disturbances of external influencing factors and complexities of internal management systems. The variation can be alleviated to a certain extent through multi-dimensional meshing, but it is restricted by the low availability of high-quality data. Meanwhile, the accurate estimation of either finer-resolution data series or tempo-spatial cumulative distributions is unachievable due to the ineffectiveness of estimation techniques, the insufficiency of data monitoring, the unpredictability of system noises, or other potential causes. As a result, properties of system components are uncertain, and the only reliable estimation may be a series of fluctuation ranges. *LP* problems in which component properties are of interval uncertainties are abbreviated as interval linear programming (*ILP*) problems.

Multiple existing programming models including *LP* can be used to characterize *ILP* problems, but the original information is misrepresented in the modeling process. For instance, when *LP* is applied to the above example, the daily generation rate has to be estimated as a representative value, e.g. the *mid value* (110 tonne/day). Regardless of what the representative value is, valuable information, e.g. boundaries of uncertain properties, is neglected in characterizing an *ILP* problem as an *LP* model. In addition, distribution-based linear programming methods (*DLP*) (Luhandjula 2014) are also available for analyzing interval uncertainties. The fluctuation range can be expressed as a random variable or a fuzzy set when distributional information (e.g. probability densities or fuzzy memberships) is constructed based on artificial assumptions. Nevertheless, the information cannot be known through reliable means, which is the original and representative feature of *ILP* problems. Imposing nonexistent distributional information to uncertain properties would result in the mismatch between real-world *ILP* problems and human-built *DLP* models.

It is desired to accurately reflect interval uncertainties of system component properties and to incorporate them into the decision-making process. To achieve it, interval linear programming (Soyster 1973; Steuer 1981; Sengupta et al. 2001; Hladík 2012) is proposed on the basis of interval analysis theories (Alefeld and Mayer 2000). Systematic introduction of *ILP* can be found in papers such as (Moore 1979) and (Sengupta and Pal 2000). As the most representative feature of *ILP* models, uncertainty in the programming objective or constraints is expressed as an interval that involves a set of real numbers within a range. In this study, the interval is named as an *interval set*. Related definitions are presented as follows.

Definition 1.1 An *interval set* ($x^{\pm}$) is a closed and bounded set of real numbers, of which the distributional information is unknown for any real number in the set.

Definition 1.2 An *interval matrix* is a matrix whose elements are *interval sets*, e.g. $\boldsymbol{Y}^{\pm} = \{y_{ij}^{\pm}\}^{m \times n}$ where $y_{ij}^{\pm}$ are *interval sets* for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Definition 1.3 An *interval vector* is a one-dimensional *interval matrix*, e.g. $\boldsymbol{X}^{\pm} = (x_1^{\pm}, x_2^{\pm}, \ldots, x_n^{\pm})$ where $x_i^{\pm}$ are *interval sets* for $i = 1, 2, \ldots, n$.

Definition 1.4 For an *interval set* ($x^{\pm}$), the *mid value* $Mid(x^{\pm}) = (x^- + x^+)/2$.

Definition 1.5 Any real number in *interval set* $x^{\pm}$, e.g. x^- or x^+, is a *whitened value* of $x^{\pm}$. (For the convenience of expressions, the event an *interval set* ($x^{\pm}$) is *whitened* as a real number (x) is denoted as $x^{\pm} \simeq x$ in this study).

Through replacing real-valued coefficients in *LP* models with *interval sets*, an *ILP* model can be generalized as maximization of $\boldsymbol{C}^{\pm}\boldsymbol{X}$ subject to $\boldsymbol{A}^{\pm}\boldsymbol{X} \le \boldsymbol{b}^{\pm}$ and $\boldsymbol{X} \ge 0$ where $\boldsymbol{C}^{\pm} = \{c_j^{\pm}\}_{1 \times n}$, $\boldsymbol{A}^{\pm} = \{a_{ij}^{\pm}\}_{m \times n}$, $\boldsymbol{b}^{\pm} = \{b_i^{\pm}\}_{m \times 1}$, $\boldsymbol{C}^{\pm}$, $\boldsymbol{A}^{\pm}$ and $\boldsymbol{b}^{\pm}$ are interval matrices or vectors, $\boldsymbol{X} = \{x_j\}_{n \times 1}$ is a vector of real-valued decision variables x_j ($j = 1, 2, \ldots, n$), and coefficients $c_j^{\pm}$, $a_{ij}^{\pm}$ and $b_i^{\pm}$ ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$) are *interval sets* that are independent with each other.

3 Model development

Interval-set coefficients $c_j^{\pm}$, $a_{ij}^{\pm}$ and $b_i^{\pm}$ ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) in *ILP* models are independent of each other, which governs effectiveness of existing *ILP* solution methods for *RESM*. However, this independence may be invalid for real-world *ILP* problems due to a multiplicity of compositions of system costs or profits. Coefficients in the objective function of *ILP* models may be of synchronic interval uncertainties, performing as the intersection of *interval sets* among coefficients.

For instance, the objective function of an *ILP* model is to maximize $f^{\pm} = c_1^{\pm} \cdot x_1 + c_2^{\pm} \cdot x_2 = (-50 \cdot d_1^{\pm} - 16 \cdot d_2^{\pm}) \cdot x_1 + (50 \cdot d_1^{\pm} + 21 \cdot d_3^{\pm}) \cdot x_2$ where $d_1^{\pm} = [1, 8]$, $d_2^{\pm} = [3, 9]$, $d_3^{\pm} = [5, 10]$ and x_1 and x_2 are non-negative decision variables. If both $d_1^{\pm}$ in $c_1^{\pm}$ and $c_2^{\pm}$ are assumed to be independent, $c_1^{\pm} = [-544, -98]$ and $c_2^{\pm} = [155, 610]$. For any solution (x_1, x_2), the objective function value ranges from $-544 \cdot x_1 + 155 \cdot x_2$ to $-98 \cdot x_1 + 610 \cdot x_2$. On the other hand, $f^{\pm} = 50 \cdot d_1^{\pm} \cdot (x_2 - x_1) - 16 \cdot d_2^{\pm} \cdot x_1 + 21 \cdot d_3^{\pm} \cdot x_2$ and $-16 \cdot d_2^{\pm} \cdot x_1 + 21 \cdot d_3^{\pm} \cdot x_2 = [-144 \cdot x_1 + 105 \cdot x_2, -48 \cdot x_1 + 210 \cdot x_2]$. If $x_2 > x_1$, we have $f^{\pm} = [50 \cdot (x_2 - x_1) - 144 \cdot x_1 + 105 \cdot x_2, 400 \cdot (x_2 - x_1) - 48 \cdot x_1 + 210 \cdot x_2] = [-194 \cdot x_1 + 155 \cdot x_2, -448 \cdot x_1 + 610 \cdot x_2]$; otherwise, $f^{\pm} = [-544 \cdot x_1 + 505 \cdot x_2, -98 \cdot x_1 + 260 \cdot x_2]$. In comparison with $[-544 \cdot x_1 + 155 \cdot x_2, -98 \cdot x_1 + 610 \cdot x_2]$, both

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ranges $[-194{\cdot}x_1+155{\cdot}x_2, -448{\cdot}x_1+610{\cdot}x_2]$ and $[-544{\cdot}x_1+505{\cdot}x_2, -98{\cdot}x_1+260{\cdot}x_2]$ are contracted since the consideration of synchronic interval uncertainties. Synchronism of *interval sets* $d_k^{\pm}$ ($k = 1, 2, 3$) complicates the process of estimating ranges of the objective function and leads to contraction of value ranges.

Neglecting synchronic effects of interval uncertainties in the process of characterizing programming systems may lead to misrepresentation of system characteristics, unacceptability of solution feasibilities, ineffectiveness of resource allocations, sacrifice of system profits, and unreliability of the decision support process. To mitigate this challenge, an explicit expression of the mapping from *interval sets* to interval-set coefficients is required for reflecting the intersection of *interval sets* and synchronisms of interval-set coefficients in the objective function. *Interval functions* as defined below are introduced to refine the relationship between *interval sets* and coefficients.

Definition 2 An *interval function* $g(\cdot)$ is a mapping from independent *interval sets* $d_k^{\pm}$ ($k = 1, 2, \ldots, r$) to a dependent *interval set* $c^{\pm}$ through interval arithmetic.

Let it be expressed as $c^{\pm} = g(d_1^{\pm}, d_2^{\pm}, \ldots, d_r^{\pm})$ where coefficients of $g(\cdot)$ are real numbers. It can be of various forms, e.g., (1) $\sum_{k\in\{1,2,\ldots,r\}}(a_k{\cdot}d_k^{\pm})$, (2) $\sum_{k\in\{1,2,\ldots,r\}}(a_k{\cdot}(-d_k^{\pm})^{Pk})$, or 3) $\sum_{k\in\{1,2,\ldots,r-1\}}\sum_{l\in\{k+1,k+2,\ldots,r\}}(a_k{\cdot}d_k^{\pm}{\cdot}d_l^{\pm})$ where a_k ($k = 1, 2, \ldots, r$) are real numbers and where P_k ($k = 1, 2, \ldots, r$) are integers. Coefficient $c^{\pm}$ ranges from $\{\min(g(d_1, d_2, \ldots, d_r)) \mid d_k^- \le d_k \le d_k^+ \text{ for } k = 1, 2, \ldots, r\}$ to $\{\max(g(d_1, d_2, \ldots, d_r)) \mid d_k^- \le d_k \le d_k^+ \text{ for } k = 1, 2, \ldots, r\}$.

Through expressing synchronic interval-set coefficients $c_j^{\pm}$ ($j = 1, 2, \ldots, n$) in the objective function of *ILP* models as *interval functions*, a synchronic interval linear programming (*SILP*) model is proposed as follows.

$$\text{Max} f^{\pm} = \boldsymbol{C}^{\pm}\boldsymbol{X} = \mathbf{G}(d_1^{\pm}, d_2^{\pm}, \ldots, d_r^{\pm})\boldsymbol{X} + \boldsymbol{HX} \quad (1.1)$$

$$\text{s.t.} \quad \boldsymbol{A}_i^{\pm}\boldsymbol{X} = \sum_{j=1,2,\ldots,n} a_{ij}^{\pm}\cdot x_j \le b_i^{\pm}, \quad i = 1, 2, \ldots, t(t \le m), \quad (1.2)$$

$$\left(b_i^+ - b_i^-\right) + \sum_{j=1,2,\ldots,n}\left(a_{ij}^+ - a_{ij}^-\right) > 0, \quad i = 1,2,\ldots,t(t \le m), \quad (1.3)$$

$$\boldsymbol{A}_i\boldsymbol{X} = \sum_{j=1,2,\ldots,n} a_{ij}\cdot x_j \le b_i, \quad i = t+1, t+2, \ldots, m, \quad (1.4)$$

$$\boldsymbol{X} \ge 0, \quad (1.5)$$

where $\boldsymbol{X} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is a vector of decision variables; $\boldsymbol{C}^{\pm} \in \{\mathbf{R}^{\pm}\}^{1\times n}$; $d_1^{\pm}, d_2^{\pm}, \ldots,$ and $d_r^{\pm}$ are n independent *interval sets* where $d_k^{\pm} = [d_k^-, d_k^+]$ and $d_k^+ > d_k^-$ for any $k \in \{1, 2, \ldots, r\}$; $\mathbf{G}\,(d_1^{\pm}, d_2^{\pm}, \ldots, d_r^{\pm}) = \{\{g_j(d_1^{\pm}, d_2^{\pm}, \ldots, d_r^{\pm})\}^{1\times n} | g_j(\cdot)$ is an *interval function* of $d_1^{\pm}, d_2^{\pm}, \ldots,$ and $d_r^{\pm}$, for $j = 1, 2, \ldots, n\}$; $\boldsymbol{H} = (h_1, h_2, \ldots, h_n)$ is a vector of real numbers; and $b_i^- \le b_i^+$ and $a_{ij}^- \le a_{ij}^+$ for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, t$.

4 Constraints transformation

Coefficients in constraints (1.4) and (1.5) are real numbers and can be effectively handled by *LP* methods (Dantzig 1947, 1963; Dantzig and Wolfe 1960). The key of constraint analysis is how to equivalently convert constraints (1.2) into crisp forms subject to constraints (1.4) and (1.5). Coefficients $a_{ij}^{\pm}$ ($i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, n$) and $b_i^{\pm}$ ($j = 1, 2, \ldots, n$) are *interval sets*. For any $i \in \{1, 2, \ldots, t\}$, they can be *whitened* as any real number between boundaries a_{ij}^- and a_{ij}^+ or b_i^- and b_i^+. Accordingly, the boundary of constrained feasible region is not as crisp as that of deterministic linear constraint ($\boldsymbol{AX} \le b$) and fluctuates between a pair of borderlines.

Definition 3 For any constraint of inequalities (1.2), the *conservative boundary* is $\boldsymbol{A}^+\boldsymbol{X} \le b^-$ and the *optimistic boundary* is $\boldsymbol{A}^-\boldsymbol{X} \le b^+$ where $\boldsymbol{A}^+ = (a_1^+, a_2^+, \ldots, a_n^+)$ and $\boldsymbol{A}^- = (a_1^-, a_2^-, \ldots, a_n^-)$.

As stated in **Lemma 1** ("Appendix"), *RESM* schemes within the *conservative boundary*, i.e. ones satisfying $\boldsymbol{A}^+\boldsymbol{X} \le b^-$, are *absolutely feasible*, and those beyond the *optimistic boundary*, i.e. ones satisfying $\boldsymbol{A}^-\boldsymbol{X} > b^+$, are infeasible. Conservative solutions satisfying inequality $\boldsymbol{A}^+\boldsymbol{X} \le b^-$ are feasible regardless of the *whitened values* of interval-set coefficients. Scheme feasibility and system security are maximized, while higher system profits are sacrificed. The solution is only desired for decision makers who would not like to take any likelihood of constraint violation. The tradeoff between system security and profit deflects to the former one excessively. On the contrary, the *optimistic boundary* corresponds to the largest feasible region, and the solution, as the most optimistic one, maximizes system profits. The solution is infeasible for almost any combination of *whitened values* of *interval sets* unless $\boldsymbol{A}^{\pm} \simeq \boldsymbol{A}^-$ and $b^{\pm} \simeq b^+$. The likelihood of constraint violation is maximized, and the tradeoff between system security and profit deflects to the latter one disproportionately.

The feasible region constrained by the *conservative boundary* can be named as the *absolutely feasible region*, while that constrained by the *optimistic boundary* be the *relatively feasible region*. They are denoted as $\boldsymbol{R}^-$ = $\{\boldsymbol{X} = (x_1, x_2, \ldots, x_n) | \boldsymbol{A}^+\boldsymbol{X} \le b^- \text{ and } \boldsymbol{X} \ge 0\}$ and $\boldsymbol{R}^+ = \{\boldsymbol{X} = (x_1, x_2, \ldots, x_n) | \boldsymbol{A}^-\boldsymbol{X} \le b^+ \text{ and } \boldsymbol{X} \ge 0\}$, respectively. The difference between the two regions is

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named as the *softly feasible region* referring to (Huang et al. 1992). Denote the *softly feasible region* as $\boldsymbol{R}_s$, and let $\boldsymbol{X}$ be any *softly feasible* solution in $\boldsymbol{R}_s$. Hence, $\boldsymbol{R}_s = \boldsymbol{R}^+ - \boldsymbol{R}^- = \{\boldsymbol{X} = (x_1, x_2, \ldots, x_n) | \boldsymbol{X} \geq 0; \boldsymbol{A}^+\boldsymbol{X} > b^-; \boldsymbol{A}^-\boldsymbol{X} \leq b^+; (b^+ - b^-) + \sum_{j=1,2,\ldots,n}(a_j^+ - a_j^-) > 0\}$.

In real-world management problems, decision makers may prefer to a balanced tradeoff between system profits and security. Solutions of uncertain feasibilities are valuable for decision making under uncertainties, at least for ones who do not want to take extremely high or low constraint-violation likelihoods. The most desired solution for decision makers who are diversified in acceptability of constraint violation risks may be in $\boldsymbol{R}_s$. To gain insight into feasibility characteristics of solutions in $\boldsymbol{R}_s$, a quantification approach is proposed based on analysis of geometric properties of the region. Let $\boldsymbol{X}$ be any *softly feasible* solution in $\boldsymbol{R}_s$.

Definition 4.1 The solution-boundary distance between $\boldsymbol{X}$ and the *conservative boundary* of $\boldsymbol{R}_s$, denoted by d_{XT}, is $|\boldsymbol{A}^+\boldsymbol{X} - b^-|/((\boldsymbol{A}^+)(\boldsymbol{A}^+)^{\mathrm{T}}) = (\boldsymbol{A}^+\boldsymbol{X} - b^-)/((\boldsymbol{A}^+)(\boldsymbol{A}^+)^{\mathrm{T}})$.

Definition 4.2 The solution-boundary distance between $\boldsymbol{X}$ and the *optimistic boundary* of $\boldsymbol{R}_s$, denoted by d_{XL}, is $|\boldsymbol{A}^-\boldsymbol{X} - b^+|/((\boldsymbol{A}^-)(\boldsymbol{A}^-)^{\mathrm{T}}) = (b^+ - \boldsymbol{A}^-\boldsymbol{X})/((\boldsymbol{A}^-)(\boldsymbol{A}^-)^{\mathrm{T}})$.

It is implied by **Proposition 1** ("Appendix") that solutions in $\boldsymbol{R}_s$ cannot be on both the *conservative boundary* and the *optimistic* one at the same time. If this does not hold feasibility of solutions around the intersection of two boundaries would be highly sensitive to fluctuations of interval coefficients. An *absolutely feasible* solution may become *absolutely infeasible* due to a tiny oscillation of coefficients. Non-intersection of the two boundaries avoids a potential issue of system security.

Suppose $\boldsymbol{X}$ is any solution in $\boldsymbol{R}_s$. As $\boldsymbol{X}$ approaches boundary $\boldsymbol{A}^+\boldsymbol{X} = b^-$, the likelihood of constraints being violated is decreased. When $\boldsymbol{A}^+\boldsymbol{X} = b^-$ or $d_{XT} = 0$, the likelihood is zero. As $\boldsymbol{X}$ approaches boundary $\boldsymbol{A}^-\boldsymbol{X} = b^+$, the likelihood is gradually increased. When $\boldsymbol{A}^-\boldsymbol{X} = b^+$, $\boldsymbol{X}$ is infeasible for all combinations of *whitened values* of coefficients $(b^\pm, \boldsymbol{A}^\pm)$ except $(b^+, \boldsymbol{A}^-)$. If $d_{XT} < 0$, we have $\boldsymbol{A}^+\boldsymbol{X} < b^-$ and $\boldsymbol{X}$ is feasible absolutely. If $d_{XL} < 0$, we have $\boldsymbol{A}^-\boldsymbol{X} > b^+$ and $\boldsymbol{X}$ is infeasible absolutely. Therefore, it is revealed that feasibility of solutions in $\boldsymbol{R}_s$ is closely related to geometric distances d_{XT} and d_{XL}. Based on the effectiveness of d_{XT} and d_{XL} at reflecting likelihoods of constraint violation, the *constraint violation likelihood* (*CVL*) is proposed to characterize the likelihood of interval-coefficient constraints ($\boldsymbol{A}^\pm\boldsymbol{X} \leq b^\pm$) being violated.

Definition 5 For constraint $A_i^\pm \boldsymbol{X} \leq b_i^\pm$ ($i = 1, 2, \ldots, t$; $t \leq m$), the *constraint violation likelihood* (CVL_i) is $d_{XT}/(d_{XT} + d_{XL})$.

From formulations of d_{XT} and d_{XL} we have $CVL_i = [(\boldsymbol{A}_i^+\boldsymbol{X} - b_i^-)/((\boldsymbol{A}_i^+)(\boldsymbol{A}_i^+)^{\mathrm{T}})]/\{[(\boldsymbol{A}_i^+\boldsymbol{X} - b_i^-)/((\boldsymbol{A}_i^+)(\boldsymbol{A}_i^+)^{\mathrm{T}})] + [(b_i^+ - \boldsymbol{A}_i^-\boldsymbol{X})/((\boldsymbol{A}_i^-)(\boldsymbol{A}_i^-)^{\mathrm{T}})]\}$ for any i. Parameter CVL_i has a series of properties that can facilitate analysis of constraint violation risks under interval uncertainties. It ranges from 0 to 1 because of non-negativity of d_{XT} and d_{XL}. It equals to zero if and only if $d_{XT} = 0$, i.e. $a_{ij}^\pm \simeq a_{ij}^+$ ($j = 1, 2, \ldots, n$) and $b_i^\pm \simeq b_i^-$. The boundary of region $\boldsymbol{R}_s$ coincides with the *conservative boundary* ($\boldsymbol{A}_i^+\boldsymbol{X} = b_i^-$). Solutions are *absolutely feasible* under intervalness. Constraints of *SILP* models would not be violated in this case. As another extreme case, it equals to one if and only if $d_{XL} = 0$, i.e. $a_{ij}^\pm \simeq a_{ij}^-$ ($j = 1, 2, \ldots, n$) and $b_i^\pm \simeq b_i^+$. The *softly feasible region* extends to the *optimistic boundary*, i.e. $\boldsymbol{A}_i^-\boldsymbol{X} = b_i^+$. Feasibility of solutions holds only for the optimistic combination of *whitened values* of coefficients, but not for others. Parameter CVL_i is a strictly monotonic increasing function of d_{XT} and is a strictly monotonic decreasing function of d_{XL}. Namely, it increases with d_{XT} and decreases with d_{XL}.

As a function of d_{XT} and d_{XL}, the functional form of CVL_i varies with multiple factors such as preferences of decision makers, the importance of and potential independence of constraints, and interrelations of coefficients. The functional form in definition 5 is only one of many alternatives. It can be in other forms that also satisfy the above properties. For example, function $d_{XT}/(d_{XT} + d_{XL})$ belongs to a family of alternative functions $g(d_{XT})/(g(d_{XT}) + g(d_{XL}))$ where $r \geq 0$, $g() \geq 0$, $g(r) = 0$ if and only if $r = 0$, and $g(r)$ increases with r. Other functions in this family can be introduced to quantify CVL_i. In this study, CVL_i is defined as a relatively simple function of d_{XT} and d_{XL} in order to ease difficulties of solving *SILP* models. Selection of optimal functional forms will be discussed in future studies.

The maximal acceptable value of CVL_i, denoted as $CVL_{i\max}$ ($i = 1, 2, \ldots, t$; $CVL_{i\max} \in [0, 1]$), is determined for all constraints in *SILP* models by decision makers in practice. Ones pursuing higher system optimality take higher $CVL_{i\max}$. Ones preferring to mitigate constraint violation risks take relatively low values of $CVL_{i\max}$ even though high system optimality is sacrificed. Under the same value of $CVL_{i\max}$, costs resulting from constraint violation may vary with constraints. A rational decision maker would determine higher $CVL_{i\max}$ for constraints of lower constraint-violation costs, and lower values for constraints of higher constraint-violation costs. Through a comprehensive consideration of all related factors, decision makers provide a group of $CVL_{i\max}$ for constraints (1.2). Note that, while parameter CVL_i is introduced to quantify the likelihood of constraint violation, parameter $CVL_{i\max}$ ($i = 1, 2, \ldots, t$) is defined for representing the maximal

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acceptable constraint-violation likelihood, i.e. the maximal values of CVL_i, for any constraint.

Theorem 1 *For any $i \in \{1, 2, \ldots, t\}$, the ith constraint in SILP model* (1) *is equivalent to*

$$\left[(1 - CVL_{imax})\left(A_i^-\right)\left(A_i^-\right)^T A_i^+ + CVL_{imax}\left(A_i^+\right)\left(A_i^+\right)^T A_i^-\right] X \le \left[(1 - CVL_{imax})\left(A_i^-\right)\left(A_i^-\right)^T b_i^- + CVL_{imax}\left(A_i^+\right)\left(A_i^+\right)^T b_i^+\right]. \quad (2)$$

Proof ("Appendix").

Denote $[(1 - CVL_{imax})(A_i^-)(A_i^-)^T A_i^+ + (CVL_{imax})(A_i^+)(A_i^+)^T A_i^-]$ and $[(CVL_{imax})(A_i^+)(A_i^+)^T b_i^+ + (1 - CVL_{imax})(A_i^-)(A_i^-)^T b_i^-]$ as $A_i(CVL_{imax})$ and $b_i(CVL_{imax})$, respectively. Inequality (2) is simplified as $A_i(CVL_{imax})X \le b_i(CVL_{imax})$. The jth ($j = 1, 2, \ldots, n$) element of vector $A_i(CVL_{imax})$, i.e. $[(1 - CVL_{imax})(A_i^-)(A_i^-)^T a_{ij}^+ + (CVL_{imax})(A_i^+)(A_i^+)^T a_{ij}^-]$, and $b_i(CVL_{imax})$ are deterministic linear functions of CVL_{imax}. They are valued as real numbers for any given CVL_{imax}. All coefficients in inequality (2) are real numbers instead of *interval sets*. Accordingly, *SILP* model (1) can be reformulated as the following model (named as *SILP-2*).

$$\mathrm{Max} f^{\pm} = C^{\pm}X = \mathbf{G}\left(d_1^{\pm}, d_2^{\pm}, \ldots, d_r^{\pm}\right)X + \mathbf{H}X \quad (3.1)$$

s.t. Inequalities (1.3)–(1.5),

$$A_i(CVL_{imax})X \le b_i(CVL_{imax}), \quad i = 1, 2, \ldots, t(t \le m), \quad (3.2)$$

Proposition 2 *For any $i \in \{1, 2, \ldots, t\}$, suppose $X \in R_s$ and CVL_{imax1} and CVL_{imax2} are any two values of CVL_{imax}. If $CVL_{imax1} < CVL_{imax2}$ and $A_i(CVL_{imax1})X \le b_i(CVL_{imax1})$, then $A_i(CVL_{imax2})X \le b_i(CVL_{imax2})$.*

Proof ("Appendix").

Remark 1 ("Appendix"): For any $CVL_{imax} \in [0, 1]$ and any $i \in \{1, 2, \ldots, t\}$, $L_{ij}(CVL_{imax}) \in [a_{ij}^-, a_{ij}^+]$ and $R_i(CVL_{imax}) \in [b_i^-, b_i^+]$ where $L_{ij}(CVL_{imax}) = [(1 - CVL_{imax}) \cdot a_{ij}^+ \cdot (A_i^-)(A_i^-)^T + (CVL_{imax}) \cdot a_{ij}^- \cdot (A_i^+)(A_i^+)^T]/[(1 - CVL_{imax}) \cdot (A_i^-)(A_i^-)^T + (CVL_{imax}) \cdot (A_i^+)(A_i^+)^T]$ and $R_i(CVL_{imax}) = [(CVL_{imax}) \cdot b_i^+ \cdot (A_i^+)(A_i^+)^T + (1 - CVL_{imax}) \cdot b_i^- \cdot (A_i^-)(A_i^-)^T]/[(1 - CVL_{imax}) \cdot (A_i^-)(A_i^-)^T + (CVL_{imax}) \cdot (A_i^+)(A_i^+)^T]$.

For the ith ($i = 1, 2, \ldots, t$) constraint ($A_i^{\pm}X \le b_i^{\pm}$) in *SILP* model (1), as CVL_{imax} increases from zero to one, constraints are violated at higher likelihoods and the boundary of the feasible region is extended. Meanwhile, the likelihood of achieving higher system optimality is enhanced. This implies that CVL_{imax} is capable of reflecting the tradeoff between constraint violation and system optimality. Does the capability hold for cases where distributional information of uncertain coefficients in constraints is known?

Under interval uncertainty, the only reliable estimation of an uncertain coefficient is a range that involves almost all potential values of the coefficient. It can be assumed that the range, which is expressed as an *interval set* in *SILP* models, is the definition domain of the distribution function under distribution-known uncertainties such as randomness and fuzziness. Let $s^{\pm}$ be any *interval set* in constraints of *SILP* models, $\check{s}$ be the corresponding uncertain coefficient under distribution-known uncertainties, and the potential value of $\check{s}$ be denoted as s. We have $s^{\pm} \in \{a_{ij}^{\pm}$ ($i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, n$), $b_i^{\pm}$ ($j = 1, 2, \ldots, n$)} and $s^- \le s \le s^+$. Let "likelihood" be a generalized term to represent the occurrence likelihood of an event. It is equivalent to "probability" under randomness, "membership" under fuzziness, or other corresponding terms. The occurrence likelihood of event ($\check{s} = s$) is expressed as $LDF(s)$ which is the likelihood density function of uncertain coefficient $\check{s}$. It is the probability density function under randomness or the fuzzy membership function under fuzziness. The cumulative distribution of uncertain coefficient $\check{s}$ is formulated as $CDF()$. We have $CDF(s) = \int_{s^-}^{s} LDF(r)\mathrm{d}r$. Suppose s_1 and s_2 are any two potential values of $\check{s}$. If $s_1 \le s_2$, then $CDF(s_1) \le CDF(s_2)$. Ranges of function $CDF(s)$ are normalized as [0, 1]. That is, $CDF(s^-) = 0$ and $CDF(s^+) = 1$. For *interval sets* $a_{ij}^{\pm}$ ($i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, n$) and $b_i^{\pm}$ ($j = 1, 2, \ldots, n$) in *SILP* models, corresponding cumulative distribution functions under distribution-known uncertainties are expressed as $F_{ij}()$ and $G_i()$, respectively. Let the likelihood of the ith constraint being violated be denoted as $DCVL(CVL_{imax})$.

Proposition 3 *$DCVL(CVL_{imax}) = \prod_{j \in \{1,2,\ldots,n\}}\{1 - F_{ij}(L_{ij}(CVL_{imax}))\} \cdot G_i(R_i(CVL_{imax}))$ for any $CVL_{imax} \in [0, 1]$ and any $i \in \{1, 2, \ldots, t\}$.*

Theorem 2 *$DCVL(CVL_{imax}) < DCVL(CVL_{imax})$ if $CVL_{imax1} < CVL_{imax2}$ where CVL_{imax1} and CVL_{imax2} are two levels of CVL_{imax}.*

Proof ("Appendix").

As indicated in **Theorem 2**, $DCVL(CVL_{imax})$ strictly monotonically increases with CVL_{imax}. It implies that $DCVL(CVL_{imax})$ and CVL_{imax} are synergistic at quantification of *constraint violation likelihoods*. Regardless of distributional information of uncertain coefficients in constraints, CVL_{imax} is effective at reflecting constraint violation under uncertainties. It is also helpful for analyzing tradeoffs between constraint violation and system optimum, which is independent with distributions of uncertainties. This is because CVL_{imax} is proposed based on analysis of geometric properties of the *softly feasible region* of *SILP* models under interval uncertainty.

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Fluctuation ranges of uncertain coefficients and thus the geometric properties do not vary with distributional information.

5 Objective-function transformation

The objective function of *SILP-2* model (3) is to maximize $\mathbf{G}(d_1^{\pm}, d_2^{\pm}, \ldots, d_r^{\pm})\boldsymbol{X} + \mathbf{H}\boldsymbol{X}$ where $\mathbf{G}(d_1^{\pm}, d_2^{\pm}, \ldots, d_r^{\pm}) = \{\{g_j(d_1^{\pm}, d_2^{\pm}, \ldots, d_r^{\pm})\}^{1\times n} | g_j()$ are *interval functions* of independent *interval sets* $d_k^{\pm}$ $(k = 1, 2, \ldots, r)$ for any $j \in \{1, 2, \ldots, n\}\}$ and where $\mathbf{H}$ is a vector of real numbers. Suppose d_k is any *whitened value* of *interval set* $d_k^{\pm}$ for any $k \in \{1, 2, \ldots, r\}$, i.e. $d_k^{\pm} \simeq d_k$ and $d_k \in [d_k^-, d_k^+]$ where $k = 1, 2, \ldots, r$. For any combination of d_k $(k = 1, 2, \ldots, r)$, the objective function is a real-valued function, $\mathbf{G}(d_1, d_2, \ldots, d_r)\boldsymbol{X} + \mathbf{H}\boldsymbol{X}$, with respect to $\boldsymbol{X}$. *SILP-2* model (3) is transformed to a deterministic linear programming problem. The optimal solution can be obtained through the simplex method (Dantzig 1947, 1963; Dantzig and Wolfe 1960). Its optimality depends on d_k $(k = 1, 2, \ldots, r)$. For any combination of d_k $(k = 1, 2, \ldots, r)$, the solution is optimal when $d_k^{\pm} \simeq d_k$ $(k = 1, 2, \ldots, r)$, but hardly remains to be the optimal one when $d_k^{\pm} \simeq d_k'$ $(d_k' \in [d_k^-, d_k^+]$; $d_k' \neq d_k$; $k = 1, 2, \ldots, r)$. Thus, it is denoted as $\boldsymbol{X}^{\mathrm{opt}}(d_1, d_2, \ldots, d_r)$ that can reflect the dependency of solution optimality on *whitened values* of *interval sets*, and is named as a *locally optimal solution*. The optimal value of objective function equals to $\mathbf{G}(d_1, d_2, \ldots, d_r)\boldsymbol{X}^{\mathrm{opt}}(d_1, d_2, \ldots, d_r) + \mathbf{H}\boldsymbol{X}^{\mathrm{opt}}(d_1, d_2, \ldots, d_r)$. It also depends on d_k $(k = 1, 2, \ldots, r)$, so it is denoted as $f^{\mathrm{opt}}(d_1, d_2, \ldots, d_r)$ and named as a *locally optimal objective value*.

The objective of this section is to identify a solution that can achieve overall optimality of objective function (3.1) under synchronic interval uncertainties. Let it be denoted as $\boldsymbol{X}_{\mathrm{opt}}$ where $\boldsymbol{X}_{\mathrm{opt}} = (x_{1\mathrm{opt}}, x_{2\mathrm{opt}}, \ldots, x_{n\mathrm{opt}})$. Given solution $\boldsymbol{X}_{\mathrm{opt}}$ and any combination of d_k $(k = 1, 2, \ldots, r)$, the objective function value, denoted as $f_{\mathrm{opt}}(d_1, d_2, \ldots, d_r)$, equals to $(\mathbf{G}(d_1, d_2, \ldots, d_r) + \mathbf{H})\,\boldsymbol{X}_{\mathrm{opt}}$. Value of $f_{\mathrm{opt}}(d_1, d_2, \ldots, d_r)$ only depends on d_k $(k = 1, 2, \ldots, r)$, which does not hold for $f^{\mathrm{opt}}(d_1, d_2, \ldots, d_r)$ that varies with both d_k $(k = 1, 2, \ldots, r)$ and $\boldsymbol{X}^{\mathrm{opt}}(d_1, d_2, \ldots, d_r)$.

Remark 2 $f^{\mathrm{opt}}(d_1, d_2, \ldots, d_r) \geq f_{\mathrm{opt}}(d_1, d_2, \ldots, d_r)$ for any combination of d_k $(k = 1, 2, \ldots, r)$.

Since $\boldsymbol{X}^{\mathrm{opt}}(d_1, d_2, \ldots, d_r)$ is the optimal solution of *SILP-2* model (3) when $d_k^{\pm} \simeq d_k$ $(k = 1, 2, \ldots, r)$, objective function $(\mathbf{G}(d_1, d_2, \ldots, d_r) + \mathbf{H})\,\boldsymbol{X}$ is optimized at solution $\boldsymbol{X}^{\mathrm{opt}}(d_1, d_2, \ldots, d_r)$. We can have $(\mathbf{G}(d_1, d_2, \ldots, d_r) + \mathbf{H})\,\boldsymbol{X}^{\mathrm{opt}}(d_1, d_2, \ldots, d_r) \geq (\mathbf{G}(d_1, d_2, \ldots, d_r) + \mathbf{H})\,\boldsymbol{X}_{\mathrm{opt}}$, i.e. $f^{\mathrm{opt}}(d_1, d_2, \ldots, d_r) \geq f_{\mathrm{opt}}(d_1, d_2, \ldots, d_r)$. It implies that, in comparison with desired solution $\boldsymbol{X}_{\mathrm{opt}}$ that does not vary with *whitened values* of *interval sets*, the *locally optimal solution* $(\boldsymbol{X}^{\mathrm{opt}}(d_1, d_2, \ldots, d_r))$ is more capable of optimizing the objective function for any combination of d_k $(k = 1, 2, \ldots, r)$.

Let $f^{\mathrm{opt}}(d_1, d_2, \ldots, d_r) - f_{\mathrm{opt}}(d_1, d_2, \ldots, d_r)$ be denoted as $L(d_1, d_2, \ldots, d_r)$. It represents the loss of local optimality at $(d_1, d_2, \ldots, d_r)$ given solution $\boldsymbol{X}_{\mathrm{opt}}$. Based on **Remark 2**, we can have $L(d_1, d_2, \ldots, d_r) \geq 0$ for any combination of d_k $(k = 1, 2, \ldots, r)$.

The overall loss of local optimality, as d_k fluctuates within $[d_k^-, d_k^+]$ $(k = 1, 2, \ldots, r)$, can be expressed as $\int\ldots\int L(d_1, d_2, \ldots, d_r)\, \mathrm{d}(d_1)\ldots\mathrm{d}(d_r)$ based on the technique of multidimensional integration. From independence of $\boldsymbol{X}^{\mathrm{opt}}(d_1, d_2, \ldots, d_r)$ and $\boldsymbol{X}_{\mathrm{opt}}$, we have $\int\ldots\int L(d_1, d_2, \ldots, d_r)\, \mathrm{d}(d_1)\ldots\mathrm{d}(d_r) = \int\ldots\int f^{\mathrm{opt}}(d_1, d_2, \ldots, d_r)\, \mathrm{d}(d_1)\ldots\mathrm{d}(d_r) - \int\ldots\int f_{\mathrm{opt}}(d_1, d_2, \ldots, d_r)\, \mathrm{d}(d_1)\ldots\mathrm{d}(d_r)$. The first integral in the right hand represents the overall local optimality, and the second one represents the overall optimality under $\boldsymbol{X}_{\mathrm{opt}}$. The desired solution $(\boldsymbol{X}_{\mathrm{opt}})$ can be obtained through indirectly minimizing the overall loss of local optimality or directly maximizing the overall optimality. Both methods are equivalent since it is valid in most real-world programming problems that the overall local optimality is deterministic and bounded. Conventionally, interval coefficients in the objective function are replaced with particular values in the corresponding interval ranges, implying that the system optimality under other combinations of coefficients is neglected; in comparison, the introduction of multidimensional integration can help identify an optimal solution that reflects the overall optimality of the objective function under synchronic interval uncertainties.

Let the feasible region of *SILP-2* model (3) be denoted as $\boldsymbol{R}_{SILP\text{-}2}$. We have $\boldsymbol{R}_{SILP\text{-}2} = \{\boldsymbol{X} \mid [(1 - CVL_{i\max})(\boldsymbol{A}_i^-)(\boldsymbol{A}_i^-)^{\mathrm{T}}\boldsymbol{A}_i^+ + CVL_{i\max}(\boldsymbol{A}_i^+)(\boldsymbol{A}_i^+)^{\mathrm{T}}\boldsymbol{A}_i^-]\boldsymbol{X} \leq [(1 - CVL_{i\max})(\boldsymbol{A}_i^-)(\boldsymbol{A}_i^-)^{\mathrm{T}}b_i^- + CVL_{i\max}(\boldsymbol{A}_i^+)(\boldsymbol{A}_i^+)^{\mathrm{T}}b_i^+]$ for $i = 1, 2, \ldots, t\,(t \leq m)$; $(b_i^+ - b_i^-) + \sum_{j=1,2,\ldots,n}(a_{ij}^+ - a_{ij}^-) > 0$ for $i = 1, 2, \ldots, t\,(t \leq m)$; $\boldsymbol{A}_i\boldsymbol{X} \leq b_i$ for $i = t + 1, t + 2, \ldots, m$; $\boldsymbol{X} \geq \mathbf{0}\}$.

Definition 6 *For any feasible solution of model* (3), *it is an integrally optimal solution if* $\int\ldots\int f_{\mathrm{opt}}(d_1, d_2, \ldots, d_r)\, \mathrm{d}(d_1)\ldots\mathrm{d}(d_r)$ *is maximized.*

Remark 3 For *SILP-2* model (3), the *integrally optimal solution* $(\boldsymbol{X}_{\mathrm{opt}})$ is the solution of the following *LP* model.

$$\mathrm{Max}\ \sum_{j=1}^{n}\left\{\left[\int\cdots\int g_j(d_1, d_2, \ldots, d_r)\, d(d_1)\ldots d(d_r)\right] + \left[h_j\prod_{k=1}^{r}\left(d_k^+ - d_k^-\right)\right]\right\}x_j \tag{4.1}$$

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s.t. $$\left[(1-CVL_{i\max})\left(A_i^-\right)\left(A_i^-\right)^T A_i^+ + CVL_{i\max}\left(A_i^+\right)\left(A_i^+\right)^T A_i^-\right]X \leq \left[(1-CVL_{i\max})\left(A_i^-\right)\left(A_i^-\right)^T b_i^- + CVL_{i\max}\left(A_i^+\right)\left(A_i^+\right)^T b_i^+\right], \quad i = 1, 2, \ldots, t(t \leq m), \tag{4.2}$$

$$\left(b_i^+ - b_i^-\right) + \sum_{j=1,2,\ldots,n}\left(a_{ij}^+ - a_{ij}^-\right) > 0, \quad i = 1, 2, \ldots, t(t \leq m), \tag{4.3}$$

$$\boldsymbol{A}_i\mathbf{X} \leq b_i, \quad i = t+1, \quad t+2, \ldots, m, \tag{4.4}$$

$$\boldsymbol{X} \geq 0. \tag{4.5}$$

where $CVL_{i\max}$ is the *maximum constraint violation likelihood* of the *i*th (i = 1, 2, …, t; $t \leq m$) interval-coefficient constraint.

Previously, many definitions were developed to locate the overall optimal solution under interval uncertainties. The *necessarily optimal solution* of an *ILP* model where interval uncertainties only exist in coefficients ($\boldsymbol{C}$) in the objective function is proposed in (Bitran 1980). As for sub-model *SILP-2* which is a mono-objective interval linear programming problem, it is an ideal solution that can achieve overall optimality of the objective function, and can be defined as a feasible solution that there does not exist another feasible solution such that the objective function is more optimal for at least one combinations of *whitened value*s of $\boldsymbol{C}$. It is related with the *integrally optimal solution* proposed in this study.

Theorem 3 *If the necessarily optimal solution exists for SILP-2 model* (3)*, it is equivalent to the integrally optimal solution.*

Proof ("Appendix").

Objective function (4.1) can be further simplified in particular cases. For instance, if intervalness does not exist in the objective function (3.1), i.e., $r = 0$, model (4) is equivalent to $\text{Max}\left\{f = \sum_{j=1}^{n} h_j x_j | X \in R_{SILP-2}\right\}$. If $\mathbf{H} = \mathbf{0}$, the equivalent form of model (4) is $\text{Max}\left\{f = \sum_{j=1}^{n} \left[\int \ldots \int g_j(d_1, d_2, \ldots, d_r)\, d(d_1)\ldots d(d_r)\right] x_j | X \in R_{SILP-2}\right\}$. If $g_j(d_1, d_2, \ldots, d_r) = \sum_{k=1}^{r}\left(p_{jk} d_k^{\pm}\right)$ where coefficients p_{jk} are real numbers for any *j* or *k*, then the objective is to maximize $\left\{\prod_{k=1}^{r}\left[d_k^+ - d_k^-\right]\right\} \sum_{j=1}^{n}\left[h_j + \sum_{k=1}^{r} p_{jk}\left(d_k^+ + d_k^-\right)/2\right] x_j$. Under linearity of *interval functions*, it is equivalent to optimize the *mid value* of the objective function, which holds under either synchronic or non-synchronic interval uncertainties. As demonstrated in followings, solutions do not vary with synchronisms of *interval sets* in the objective function of *SILP* models when *interval functions* are linear polynomials of *interval sets*. If $g_j(d_1, d_2, \ldots, d_r) = d_j^{\pm}$ (j = 1, 2, …, n), the *SILP* model is degenerated to an *ILP* model where *interval sets* are not synchronized. Objective function (4.1) is transformed to maximize $\sum_{j=1}^{n} \left\{\left[0.5 \prod_{k=1}^{n}\left(d_k^+ - d_k^-\right)\left(d_j^+ + d_j^-\right)\right]\left[h_j \prod_{k=1}^{r}\left(d_k^+ - d_k^-\right)\right]\right\}$. Nonlinearity in a number of real-world programming problems can be decomposed as a series of polynomials through techniques of Taylor or Fourier expansions. It is possible that $g_j(d_1^{\pm}, d_2^{\pm}, \ldots, d_r^{\pm})$ is a convex combination of *interval sets* $d_k^{\pm}$ (k = 1, 2, …, r), i.e., $g_j(d_1, d_2, \ldots, d_r) = \sum_{l=1}^{sj}\left[p_{jl} \prod_{k=1}^{r}\left(d_k^{\pm}\right)^{\wedge} e_{jlk}\right]$ where e_{jlk} is an integer for any *j*, *k*, or $l \in \{1, 2, \ldots, s_j\}$. Objective function (4.1) is equivalent to maximize $\sum_{j=1}^{n}\left\{\left[\sum_{l=1}^{sj} p_{jl}\left\{\prod_{k=1}^{r}\left[\left(d_k^+\right)^{\wedge}(e_{jlk}+1) - \left(d_k^-\right)^{\wedge}(e_{jlk}+1)\right]/(e_{jlk}+1)\right]\right\} + \left[h_j \prod_{k=1}^{r}\left(d_k^+ - d_k^-\right)\right]\right\} xj$.

Coefficients of objective function in *SILP-2* model (4) are real numbers rather than *interval functions*. Model (4) is a deterministic linear programming problem. It can be effectively solved by the simplex method (Dantzig 1947, 1963; Dantzig and Wolfe 1960). The solution is $\boldsymbol{X}_{\text{opt}}$ where $\boldsymbol{X}_{\text{opt}} = (x_{1\text{opt}}, x_{2\text{opt}}, \ldots, x_{n\text{opt}})$. The maximal multidimensional integral of objective function involving synchronic interval coefficients, named as the *optimal integral*, equals to $\sum_{j=1}^{n}\left\{\left[\int \ldots \int g_j(d_1, d_2, \ldots, d_r)\, d(d_1)\ldots d(d_r)\right] + \left[h_j \prod_{k=1}^{r}\left(d_k^+ - d_k^-\right)\right]\right\} x_{j\text{opt}}$. Accordingly, objective function (1.1) ranges from {min (($\mathbf{G}$(d_1, d_2,…, d_r) + $\mathbf{H}$) $\boldsymbol{X}_{\text{opt}}$) | $d_k^- \leq d_k \leq d_k^+$ for k = 1, 2, …, r} to {max (($\mathbf{G}$(d_1, d_2,…, d_r) + $\mathbf{H}$) $\boldsymbol{X}_{\text{opt}}$) | $d_k^- \leq d_k \leq d_k^+$ for k = 1, 2, …, r}.

Based on the proposition of *constraint violation likelihoods* and *integrally optimal solutions*, a violation-constrained interval integral (*VCII*) method is explored for solving *SILP* models such as model (1). The structure and significance of this study are presented in Fig. 1. In terms of solving an *SILP* model, the proposed *VCII* method is specified as follows.

Step 1 For any $i \in \{1, 2, \ldots, t\}$, decision makers provide an acceptable *maximum constraint violation likelihood* $CVL_{i\max}$ ($0 \leq CVL_{i\max} \leq 1$) of the *i*th interval-coefficient constraint (1.2).

Step 2 Interval-coefficient constraints are transformed into deterministic linear inequalities (3.2) that do not involve interval-set coefficients.

Step 3 SILP model (1) is converted to *SILP-2* model (3) of which coefficients in the objective function are *interval functions*.

Step 4 Synchronic interval-set coefficients are *whitened* as real numbers based on the concept of the *integrally optimal solution* and the technique of multidimensional integration.

Step 5 SILP-2 model (3) is equalized as a deterministic linear programming model (4) where all coefficients are real numbers.

Step 6 Under a set of *maximum constraint violation likelihoods* $\{CVL_{i\max}\}_{i=1,2,\ldots,t}$, an *integrally optimal*

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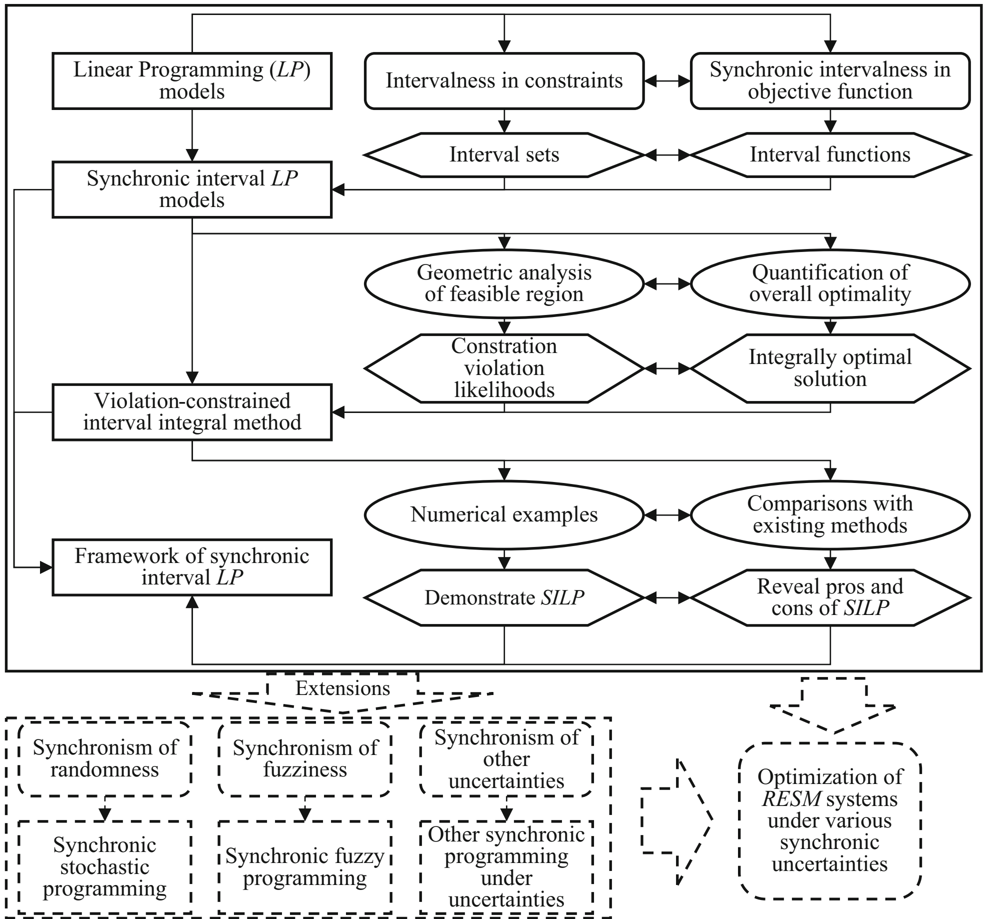

Fig. 1 The structure and significance of the developed *SILP* method

solution ($\boldsymbol{X}_{\text{opt}}$) and corresponding *optimal integral* are generated through the simplex method (Dantzig 1947, 1963; Dantzig and Wolfe 1960).

Step 7 The range of objective function (1.1) is obtained through $f_{\text{opt}}^{\pm} = \mathbf{G}(d_1^{\pm}, d_2^{\pm},\ldots, d_r^{\pm})\boldsymbol{X}_{\text{opt}} + \mathbf{H}\boldsymbol{X}_{\text{opt}}$.

6 Examplification

A simplified *RESM* problem is proposed to represent *RESM* systems involving synchronic interval uncertainties in an objective of system profit and interval uncertainties in constraints of environmental loading capacity and resources availability. This problem is provided to demonstrate advantages of the proposed *SILP* method as well as the detailed procedures of the *VCII* algorithm. Such a problem can be formulated as the following *SILP* model:

$$\text{Max} f^{\pm} = (-50 \cdot d_1^{\pm} - 16 \cdot d_2^{\pm}) \cdot x_1 + (50 \cdot d_1^{\pm} + 21 \cdot d_3^{\pm}) \cdot x_2 \tag{5.1}$$

$$\text{s.t.} \quad [-2.13, -1.91]x_1 + [0.96, 1.18]x_2 \leq [1.89, 2.10], \tag{5.2}$$

$$[-1.10, -0.94]x_1 + [2.85, 3.04]x_2 \leq [17.35, 19.91], \tag{5.3}$$

$$[0.97, 1.12]x_1 + [0.88, 1.07]x_2 \leq [15.06, 16.47], \tag{5.4}$$

$$x_1 \geq 0, x_2 \geq 0, \tag{5.5}$$

where $d_1^{\pm} = [1, 8]$, $d_2^{\pm} = [3, 9]$, and $d_3^{\pm} = [5, 10]$.

Step 1 Suppose the *maximum constraint violation likelihoods* for all constraints, denoted as $CVL_{i\max}$ where $i = 1$, 2 and 3, are equal to any element in set {0, 0.1, 0.2, …, 0.9, 1}. That is, $CVL_{i\max} = CVL_{\max}$ for any $i \in \{1, 2, 3\}$ and $v \in \{0, 0.1, 0.2, \ldots, 0.9, 1\}$.

Step 2 For any interval-coefficient constraint, generalized as $a_1^{\pm}x_1 + a_2^{\pm}x_2 \leq b^{\pm}$, it is equivalent with $(1 - CVL_{\max})[(a_1^-)^2 + (a_2^-)^2](a_1^+x_1 + a_2^+x_2) + CVL_{\max}[(a_1^+)^2 + (a_2^+)^2](a_1^-x_1 + a_2^-x_2) \leq (1 - CVL_{\max})[(a_1^-)^2 + (a_2^-)^2]b^- + CVL_{\max}[(a_1^+)^2 + (a_2^+)^2]b^+$. Correspondingly, deterministic linear constraints are

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$$(-10.426 - 0.311CVL_{\max})x_1 + (6.441 - 1.602CVL_{\max})x_2 \leq 10.317 + 0.268CVL_{\max} \quad (6.1)$$

$$(-8.773 - 2.365CVL_{\max})x_1 + (28.371 + 0.486CVL_{\max})x_2 \leq 161.919 + 39.674CVL_{\max} \quad (6.2)$$

$$(1.921 + 0.406CVL_{\max})x_1 + (1.835 + 0.276CVL_{\max})x_2 \leq 25.832 + 13.684CVL_{\max} \quad (6.3)$$

Step 3 SILP model (5) is converted to the maximization of $f^{\pm} = (d_1^{\pm} + d_2^{\pm})x_1 + (d_1^{\pm} + d_3^{\pm})x_2$ subject to inequalities (6.1), (6.2), (6.3) and (5.5). Only coefficients in the objective function are synchronic *interval functions*.

Step 4 The *integrally optimal solution* can be obtained from maximizing $\sum_{j=1}^{n}\left\{\left[\int\ldots\int g_j(d_1,d_2,\ldots,d_r)\,d(d_1)\ldots d(d_r)\right] + \left[h_j\prod_{k=1}^{r}(d_k^{+} - d_k^{-})\right]\right\}x_j$ subject to deterministic feasible region. Objective function $f^{\pm} = (d_1^{\pm} + d_2^{\pm})x_1 + (d_1^{\pm} + d_3^{\pm})x_2$ is transformed to deterministic inequality $-471870x_1 + 562275x_2$ that is equivalent with $-642x_1 + 765x_2$.

Step 5 SILP model (5) is further reduced into a deterministic linear programming model, i.e. maximization of the objective function $f = -471870x_1 + 562275x_2$ subject to inequalities (6.1), (6.2), (6.3) and (5.5), where all coefficients are real numbers.

Step 6 Corresponding to various *constraint violation likelihoods* from 0 to 1, a series of *integrally optimal solutions* ($x_{1\text{opt}}$ and $x_{2\text{opt}}$) and corresponding *optimal integral* are obtained (Table 1; Fig. 2) through the simplex method (Dantzig 1947, 1963; Dantzig and Wolfe 1960).

Step 7 Since $x_{2\text{opt}} > x_{1\text{opt}}$ for any $CVL_{\max} \in \{0, 0.1, 0.2, \ldots, 0.9, 1\}$, the lower boundary of the objective function (f_{opt}^{-}) is equal to $50\cdot(x_{2\text{opt}} - x_{1\text{opt}})\cdot d_1^{-} - 16\cdot d_2^{+}\cdot x_{1\text{opt}} + 21\cdot d_3^{-}\cdot x_{2\text{opt}}$, while the upper one ($f_{\text{opt}}^{+}$) is $50\cdot(x_{2\text{opt}} - x_{1\text{opt}})\cdot d_1^{+} - 16\cdot d_2^{-}\cdot x_{1\text{opt}} + 21\cdot d_3^{+}\cdot x_{2\text{opt}}$. In Table 1 and Fig. 1, both boundaries and the *mid value*s of the objective function are presented.

It is indicated in Table 1 that $CVL_{\max}$ is effective at quantifying the likelihood of interval constraints being violated. The integral of the objective function with synchronic interval-set coefficients is helpful for presenting the overall optimality of the objective function with respect to independent *interval sets*. As $CVL_{\max}$ increases from 0 to 1, the *optimal integral* of the interactive-interval-coefficient objective function with respect to independent *interval sets* monotonically increases from 2274673 to 3260800 (see Table 1). The lower and upper boundaries of the objective function also increase from −670.742 and 3765.536 to −184.808 and 4621.270 monotonically (see Table 1). This is mainly because $CVL_{\max}$ is defined for quantifying the likelihood at which an interval-coefficient constraint is violated. The higher the value of *constraint violation likelihoods* is, the higher the likelihood of constraint being violated is, and the broader the feasible region is. Due to the expansion of the feasible region, the value ranges of the objective function are increased. As a result, the boundaries, the *mid value* as well as the *optimal integral* of the objective function monotonically increase with *constraint violation likelihoods*.

7 Discussion

7.1 Constraint violation risks

VCII is effective at reflecting constraint violation risks under interval uncertainties. It is verified by a comparison of *VCII* and existing *ILP* methods based on transformation

Table 1 Solutions of model (5)** through violation-constrained interval integral analysis

$CVL_{i\max}$	*Optimal integral*	*F_lower*	*F_upper*	X_1	X_2	*F_mean*
0.0	3.250E+05	426.633	2668.161	3.135	6.677	1547.397
0.1	3.263E+05	429.376	2678.203	3.133	6.691	1553.789
0.2	3.276E+05	432.126	2688.256	3.131	6.706	1560.191
0.3	3.290E+05	434.884	2698.319	3.128	6.721	1566.602
0.4	3.303E+05	437.649	2708.392	3.125	6.735	1573.021
0.5	3.317E+05	440.422	2718.475	3.123	6.750	1579.449
0.6	3.330E+05	443.202	2728.568	3.120	6.765	1585.885
0.7	3.344E+05	445.989	2738.671	3.117	6.779	1592.330
0.8	3.357E+05	448.784	2748.783	3.115	6.794	1598.783
0.9	3.371E+05	451.586	2758.904	3.112	6.808	1605.245
1.0	3.385E+05	454.395	2769.035	3.109	6.823	1611.715

$CVL_{i\max}$ represents the *maximum constraint violation likelihood*; *f_lower*, *f_upper* and *f_mean* represent the lower boundary of, the upper boundary of, and the mean value of objective function under interval uncertainties, respectively

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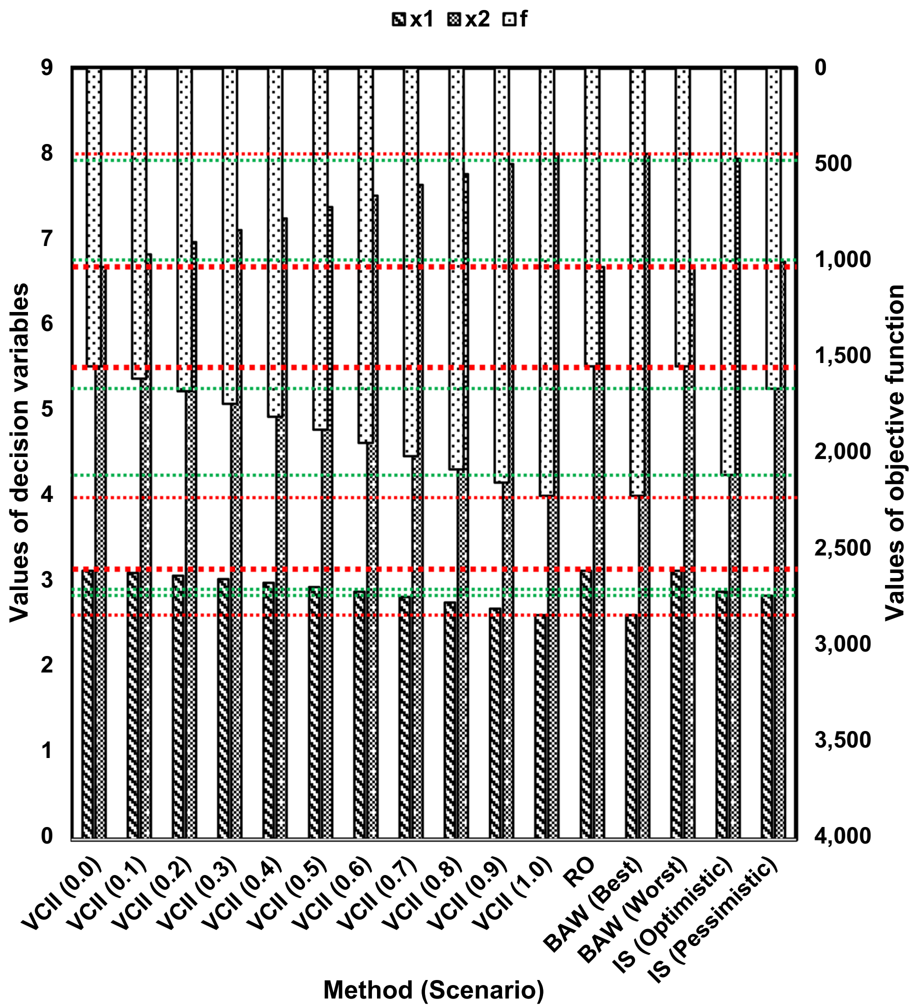

Fig. 2 Comparison of solutions under whitened interval uncertainties in the objective function

of original *SILP* model (5). To eliminate disturbances of interval uncertainties in the objective function on constraints, *interval sets* in objective function (5.1) are *whitened* as *mid values*. Objective function (5.1) is transformed as the maximization of a deterministic function. Namely, $f = (-50 \cdot d_{1\text{mid}} - 16 \cdot d_{2\text{mid}}) \cdot x_1 + (50 \cdot d_{1\text{mid}} + 21 \cdot d_{3\text{mid}}) \cdot x_2$ where $d_{1\text{mid}} = (1 + 8)/2 = 4.5$, $d_{2\text{mid}} = (3 + 9)/2 = 6$, and $d_{1\text{mid}} = (5 + 10)/2 = 7.5$. Accordingly, *SILP* model (5) is reformulated as the maximization of $f = -321 \cdot x_1 + 382.5 \cdot x_2$ subject to inequalities (5.2)–(5.5). Let the model after reformulation be denoted as *SILP-C*.

Interval uncertainties only exist in constraints (5.2)–(5.4) and lead to intervalness of decision variables and of objective function values. Many existing methods (Rommelfanger et al. 1989; Levin 1994; Maqsood and Huang 2003) excluding *VCII* are available for solving the *SILP-C* model. Among them, representative ones include the robust optimization (*RO*) (Ben-Tal et al. 2009), the best-and-worst approach (*BAW*) (Tong 1994), and the interval-solution method (*IS*) (Huang et al. 1992). Corresponding solutions are presented in Fig. 2. *RO* can generate a robust solution that is feasible for any combination of *whitened values* of *interval sets* in constraints and that is optimal for at least one combination. It is (3.135, 6.677) for the *SILP-C* model, and corresponds to a case of *VCII* in which *CVL* equals to zero. A pair of solutions that correspond to the best objective function value and the worst one can be obtained through two independent *LP* sub-models in *BAW*. For the *SILP-C* model they are (3.135, 6.677) and (2.618, 7.996) which match the lowest and the highest *constraint violation likelihoods* in *VCII*. The worst solution in *BAW*, i.e. (3.135, 6.677), is identical with the robust solution in *RO*. Through the *IS* method two sets of deterministic solutions, i.e. (2.891, 7.940) and (2.845, 6.737), can be obtained for optimistic and pessimistic decision makers, respectively. Coupling of them generates interval-set solutions ([2.845, 2.891], [6.737, 7.940]). The boundaries of decision variables x_1 and x_2 and the objective function f under interval uncertainties are contracted from [2.618, 3.135], [6.677, 7.996] and [1547.397, 2218.231] in *VCII* to [2.845, 2.891], [6.737, 7.940] and [1663.595, 2108.831] in IS for achieving continuity of decision space. The

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contraction can be achieved through adjusting values of *constraint violation likelihoods* in *VCII*. It is indicated that, for *ILP* models in which interval uncertainties only exist in constraints, solutions obtained from *RO*, *BAW* or *IS* are particular cases of *VCII*. These existing methods can only provide decision makers with schemes under particular levels of constraint violation risks. The detailed tradeoff between constraint violation risks and system optimality cannot be reflected, which is mitigated in *VCII*.

7.2 Interval uncertainties in constraints

In addition to *RO*, *BAW* and *IS*, distribution-based linear programming methods such as chance-constrained programming (*CCP*) (Cheng et al. 2009) are also available to address interval uncertainties in constraints of *ILP* models. Interval uncertainties are artificially assumed as normally distributed random variables in *CCP*. Distributional functions of uncertain properties are constructed based on the theory that, for standard normal distributions, approximately 99.75% of sample values fall within three standard deviations of the mean. The *SILP-C* model is randomized as a stochastic *LP* model in which coefficients in both sides of constraints are random variables. Let the model be denoted as *SILP-R*. Constraints in the randomized *ILP* model is transformed to deterministic inequalities given acceptable levels of constraint violation probabilities (p_i). The simplex method is employed to obtain solutions under a series of p_i. As for the *SILP-R* model, all solutions obtained from *CCP* are shown in Fig. 3. It is indicated that the tradeoff between constraint violation risks and system optimality can be reflected in CCP as *VCII* does. *CCP* can provide distributional information of constraint violations, which is unachievable for *VCII*. However, the reliability of

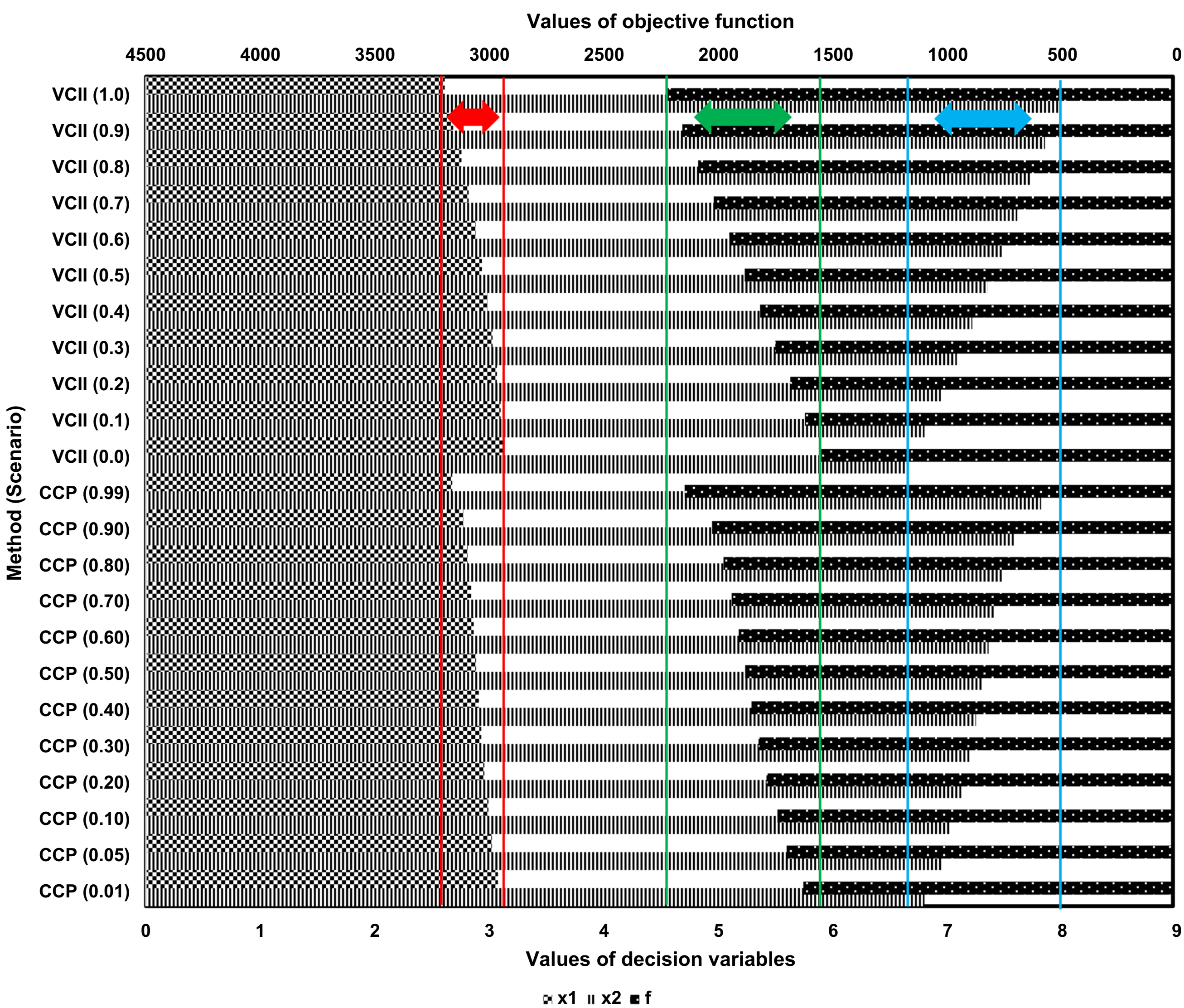

Fig. 3 Comparison of solutions of *SILP-R* model through violation-constrained interval integral analysis and chance-constrained programming

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CCP is only accepted under ideal conditions. That is, sufficient evidence can support the assumption that uncertain properties are normally distributed. Besides, the assumed random distributions deviate from original interval uncertainties. These distributions do not exist and are artificially added. Furthermore, the *VCII* method can be an alternative of *CCP* if these assumptions are acceptable in particular cases, which was stated in **Theorem 2**. As shown in Fig. 3, ranges of decision variables x_1 and x_2 and objective function *f* in the *SILP-R* model are [2.683, 3.002], [7.048, 7.845] and [1732.310, 2139.473], respectively, as p_i varies within [0.01, 0.99]. All of them are contracted in comparison with *VCII* solutions. Although these ranges can be enlarged in *CCP* through expanding the value range of p_i, e.g. from [0.01, 0.99] to [0.0001, 0.9999], the reliability of normal-distribution assumptions decreases sharply as p_i approaches boundaries (0 and 1). It is because relatively low sample sizes and high instabilities of uncertain properties at tails of normal distributions may result in low distribution estimation efficiencies. Under almost any combination of constraint violation probabilities, the corresponding *CCP* solutions can be obtained through *VCII*. In addition, *VCII* does not rely on distributional forms, while *CCP* does. It is particularly meaningful when distributions of uncertain properties are too irregular or unstable to be precisely formulated. In these cases, *VCII* is more reliable than *CCP* for decision making under uncertainties.

7.3 Overall optimality of objective function

The proposed *VCII* method is capable of achieving overall optimality of the objective function in *SILP* models under interval uncertainties. To clarify it, a comparison between *VCII* and existing *ILP* methods (e.g., *BAW*, *RO* and *IS*) is conducted based on the transformation of original *SILP* model (5). *Interval sets* in constraints (5.2)–(5.4) are *whitened* as *mid values* for eliminating influences of interval-set synchronisms on the objective function. Interval-coefficient constraints (5.2)–(5.4) are transformed as deterministic inequalities (7.2)–(7.4). Meanwhile, *interval functions* in the objective function are replaced with corresponding value ranges under independence of *interval sets*. Accordingly, *SILP* model (5) is transferred to an *ILP* model as follows where *interval sets* only exist in the objective function.

$$\text{Max}\, f = [-544, -98] \cdot x_1 + [155, 610] \cdot x_2 \tag{7.1}$$

$$\text{s.t.} - 2.02x_1 + 1.07x_2 \le 1.995, \tag{7.2}$$

$$-1.02x_1 + 2.945x_2 \le 18.63, \tag{7.3}$$

$$1.045x_1 + 0.975x_2 \le 15.765, \tag{7.4}$$

$$x_1 \ge 0,\ x_2 \ge 0. \tag{7.5}$$

The solutions of model (7) are presented in Fig. 4. The optimal solution is (2.894, 7.328) when *VCII* is employed.

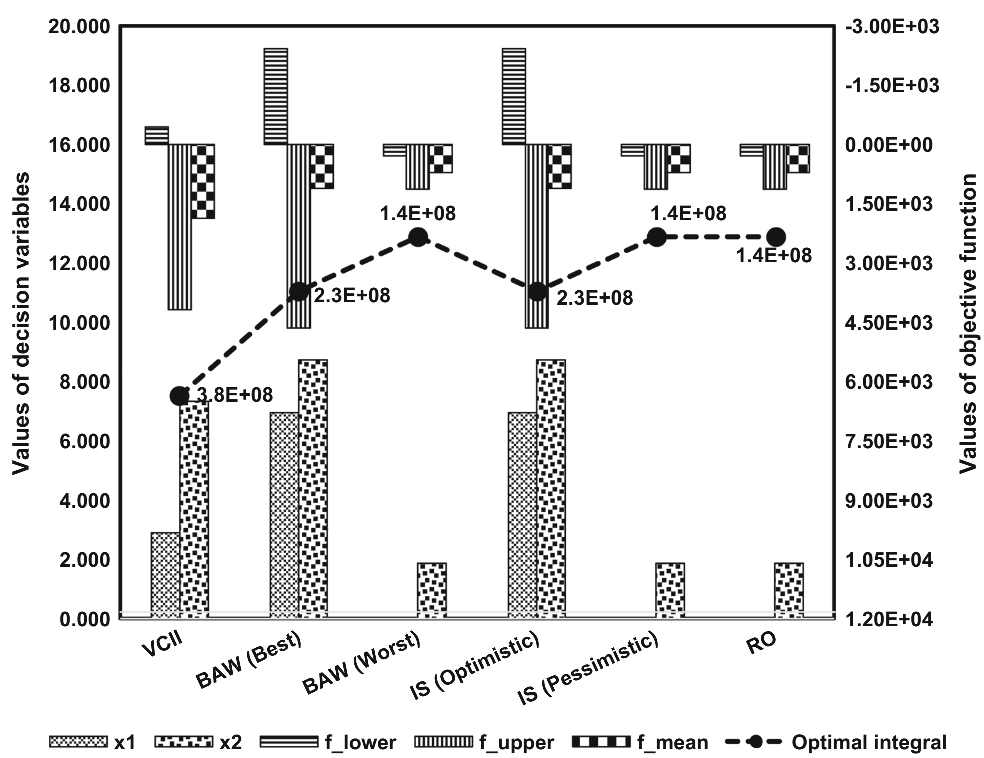

Fig. 4 Solutions of model (7) under *whitened* interval uncertainties in constraints. (*f_lower*, *f_upper* and *f_mean* represent the lower boundary of, the upper boundary of, and the mean value of objective function under interval uncertainties, respectively)

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Correspondingly, the objective function ranges from −438.571 to 4186.687, and the *integrally optimal solution* equals to 3.803E+08 under interval uncertainties of objective-function coefficients. Solutions are identical for *BAW* and *IS*, and *RO* corresponds to the worst scenario in BAW or the pessimistic scenario in *IS*. The *integrally optimal solution* reflects the overall optimality of the objective function as coefficients fluctuate within *interval sets*. In comparison with *VCII*, the *integrally optimal solution* in *BAW*, *IS* and *RO* is significantly decreased. The *mid value* of objective function (7.1) in *VCII* is higher than that in any existing *ILP* methods, which also reflects the effectiveness of *VCII* at achieving overall optimality. Through construction of two sub-models in existing *ILP* methods two sets of solutions are obtained, i.e. (6.941, 8.730) and (0, 1.864). The former sub-model focuses on the upper bound of objective function, while the latter one on the lower bound. For either of cases the aim is to realize local optimality rather than overall optimality. As a result, the upper bound of (7.1) reaches 4645.069 at the cost of decreased lower-bound values. The upper bound is sacrificed, decreasing from 4645.069 to 1137.336, for maximizing the lower bound. Thus, it is indicated that *VCII* is effective at achieving overall optimality of the objective function in *ILP* models.

7.4 Synchronism of *interval sets*

Synchronisms of *interval sets* in the objective function of *SILP* models can be addressed through the proposed *VCII* method. To analyze such an effect, a comparison between *VCII* and existing *ILP* methods is conducted on the basis of a modification of *SILP* model (5). *Interval sets* in constraints (5.2)–(5.4) are *whitened* as *mid values* for eradicating disturbances of interval uncertainties in constraints. Objective function (5.1) remains. The modified model can be formulated as follows.

$$\text{Max}\, f^{\pm} = (-50 \cdot d_1^{\pm} - 16 \cdot d_2^{\pm}) \cdot x_1 + (50 \cdot d_1^{\pm} + 21 \cdot d_3^{\pm}) \cdot x_2 \qquad (8.1)$$

$$\text{s.t.Inequalities } (7-2) \sim (7-5). \qquad (8.2)$$

The synchronism of *interval sets* $d_k^{\pm}$ $(k = 1, 2, 3)$ in the objective function is neglected in one case, while not in another case. *VCII* is employed to solve model (8) in both cases. Solutions are presented in Table 2. In the first case, the synchronic *interval set* $(d_1^{\pm})$ is taken as two unsynchronized ones. As a result, four *interval sets* coexist in the objective function (8.1). The solution is (2.894, 7.328), the objective function value ranges from −438.571 to 4186.687, and the *integrally optimal solution* is 2.755E+06. In the second case, the synchronic effect of *interval set* $(d_1^{\pm})$ is kept. The solution from *VCII* does not change because of linearity of *interval functions* in the objective function (5.1) or (8.1). As a general rule, the synchronism of *interval sets* in the objective function of *SILP* models does not affect *VCII* solutions when *interval functions* in the objective function are linear polynomials of synchronic *interval sets*. This rule can hardly hold if *interval functions* are nonlinear, especially high-order, functions of synchronic *interval sets*. In the second case, the boundary of the objective function (8.1) is [574.417, 3173.699] and the *integrally optimal solution* is 3.936E+05. When the synchronism is neglected the region in which uncertain coefficients fluctuate would be enlarged. Even given the same solution, the boundary of the objective function would be expanded under interval uncertainties. The *integrally optimal solution*, which reflects the overall optimality of the objective function under interval uncertainties of coefficients, would be increased. As for model (8), the boundary expands to [−438.571, 4186.687] and the *integrally optimal solution* increases to 2.755E+06. Nevertheless, the expanded part of the boundary, i.e. [−438.571, 574.417) and (3173.699, 4186.687] for model (8), is unachievable due to synchronism of *interval sets* in the objective function. Neglecting synchronism of *interval sets* in the objective function of *SILP* models, as existing *ILP* methods do, would lead to significant deviations in the estimation of programming objective boundaries under intervalness. No matter whether the programming objective is to achieve the maximum or the minimum the lower boundary would be underestimated and the upper one would be overestimated. The deviated boundary implies incorrect scientific support to decision makers, and additional costs may be required to tackle with unreliable management schemes. Under nonlinearities of *interval functions*, the obtained solutions may be completely infeasible for constraints and lead to abnormally

Table 2 Effects of synchronism on solutions of model (8)

Case	*F_lower*	*F_upper*	X_1	X_2	*Optimal integral*
VCII(Syn)	574.417	3173.699	2.894	7.328	3.936E+05
VCII(Unsyn)	−438.571	4186.687	2.894	7.328	2.755E+06

f_lower and *f_upper* are the lower boundary of and the upper boundary of the objective function, respectively; *VCII* represents the method of violation-constrained interval integral analysis; Synchronisms of interval uncertainties are taken into account in *VCII*(Syn), while not in *VCII*(Unsyn)

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high penalties due to the neglecting of such an effect in existing *ILP* methods.

7.5 Comparisons with existing methods

SILP problems are extensions of *LP* problems. Extended features include interval uncertainties of system component properties and synchronic effects of these uncertainties. For *SILP* problems there are many alternative decision support techniques. In addition to the *SILP* method proposed in this study, representative ones consist of *LP*, *RO*, *ILP* (e.g. *BAW* and *IS*) and *DLP* (e.g. *CCP* and *FRP*). In the process of modeling *SILP* problems, interval uncertainties are misinterpreted in both *LP* and *DP*. Fluctuation ranges of uncertain properties are represented as averaged values in *LP*, leading to the missing of original information that is valuable. In *DP*, distributional information of uncertain properties, which is originally unknown due to lack of reliable data and technical support, is imposed based on artificial assumptions. Irrespective of original information of interval uncertainties in *SILP* problems, the measure of either *LP* or *DLP* can be hardly replied on. In contrast, *RO* and *ILP* are capable of reflecting these uncertainties. However, very few studies in *RO* and *ILP* focus on characterization of synchronic effects of interval uncertainties. Failure in incorporating interval uncertainties and their synchronisms into the modeling process would significantly decrease the reliability of decision support efforts. This challenge is mitigated through the proposition of *interval functions* in *SILP* models.

In terms of solving *SILP* models, existing methods such as Monte Carlo Simulation (*MCS*), *LP*, *RO*, *BAW*, *IS* and *CCP* are not reliable due to failure in reflecting realities. Even though these methods can be enforced to do it, the solving process is criticized in many aspects. *MCS* relies on distributions of uncertain coefficients. They are unknown for *SILP* models, which is one of the reasons that *ILP* and *SILP* models emerge. Heavy computational loads also restrict the applicability of *MCS* for large-scale programming problems. Under interval uncertainties, boundaries of decision variables and of objective function values cannot be provided in *LP*. Only a few of specific decision schemes, e.g. the robust one in *RO*, the worst and best ones in *BAW* and the optimistic and pessimistic ones in IS, are focused on the methods of *RO*, *BAW* and *IS*. These schemes may be acceptable for *DM*s under particular (especially extreme) conditions, but cannot be desired for all *DM*s who are diversified in acceptability of constraint violation risks and in expectation of system optimality. The overall optimality of the objective function under interval uncertainties is unachievable for existing methods. Moreover, synchronism of *interval sets* in the objective function is neglected in the solving process. Schemes may deviate far away from desired ones. It is of high likeliness that the system profit is over- or under-estimated, which may misdirect decision makers. Execution of corresponding schemes may result in a variety of challenges in real-world cases such as ineffective allocation of resources, a decrease of system optimality, violation of constraints, and unexpectedly high costs of recourse measures. These challenges are overcome to a certain extent in the developed *VCII* method.

To sum up, comparisons between *SILP* and selected existing methods in this study reveal the effectiveness of *SILP* in multiple aspects. In addition to interval uncertainties of component properties in linear programming systems, synchronic effects of interval uncertainties in the programming objective can also be reflected through proposition of *interval function*s in *SILP* models. Since geometric features of feasible regions under interval uncertainties are independent with distributions of coefficients, *constraint violation likelihood*s are proposed for quantifying violation of constraints based on the geometric analysis. Analysis of the tradeoff between *constraint violation likelihood*s and system optimality is enabled in the developed *VCII* algorithm. Significant potential negative influences of synchronic interval uncertainties, which may lead to ineffective allocation of resources, a decrease of system optimality, violation of constraints and unexpectedly high costs of recourse measures in real-world problems, are disclosed based on numerical examples and are mitigated in *VCII*. The proposition of multidimensional interval integration in *VCII* achieves the overall optimality of *SILP* problems under synchronic interval uncertainties. The provision of *integrally optimal solution*s associated with a variety of *constraint violation likelihood*s facilitates risk analysis and management of *SILP* problems.

Furthermore, the *SILP* model and the *VCII* method exploit a framework of addressing synchronisms of uncertainties in programming problems. As for other types of uncertainties, e.g. randomness or fuzziness, the synchronic effect also exists in real-world management problems. It is a particular case of interactions or dependence of these uncertainties. The representative feature is that this interaction is easily identified. Depending on the framework of this study, synchronisms of randomness or fuzziness can be refined as functional forms, and be incorporated into the optimization process. Based on constraint violation analysis, multidimensional integration of synchronic probabilistic or possibilistic distributions is helpful for providing decision makers with the overall optimal management schemes. A series of disasters due to neglecting synchronisms of intervalness, randomness or fuzziness in decision support processes can be mitigated, which is the most significant contribution of this study.

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7.6 Potential extensions of *SILP*

The proposed *SILP* method in this study is the first attempt to integrate *interval function*s, geometric analysis, and interval integral together for providing reliable decision support for *RESM* under synchronic interval uncertainties. This method is only feasible for *RESM* problems that can be formulated as *LP* models with synchronic interval coefficients, and is deficient in some aspects including but not limited to the followings. The feasibility of the developed method for *SILP* problems depends on one condition that system component properties are uncertain with unknown distributions. For problems in which uncertainties do not exist or distributional information is known, *SILP* cannot be the most desired decision support tool. *SILP* models are helpful for addressing synchronic interval-set coefficients in the objective function. Synchronisms of interval, random or fuzzy coefficients in constraints and the objective function cannot be reflected if they exist. In *VCII*, acceptable *constraint violation likelihood*s of interval-coefficient constraints are determined through subjective judgments. A quantitative method is lacked to avoid incorrectness in this process. The multidimensional integral of the objective function under synchronic interval uncertainties may not be equivalent to deterministic linear functions due to nonlinearities of *interval function*s. Interval uncertainties in both constraints and the objective function of *SILP* models are addressed separately in *VCII*. Potential dependency among them may be neglected, which may affect obtained solutions and their feasibilities. Real-world *SILP* problems may involve other complexities such as diversity of uncertainties, a multiplicity of programming objectives, discreteness of component properties, dynamics of system features, the nonlinearity of component interrelationships, and interactions of them. For any of these cases, the developed *SILP* method is inapplicable.

To enhance the feasibilities of *SILP* in mitigating the aforementioned challenges in *RESM*, efforts can be made to improve or extend *SILP* in many aspects. For instance, synchronic probabilistic or possibilistic programming models, as variants of *SILP* models, are proposed for addressing synchronisms of randomness or fuzziness in real-world *RESM* or other planning problems; in these models, coefficients are represented as functions of random variables or fuzzy sets to reflect the synchronic effects in these coefficients under diversities and interactions of *RESM* system components; to solve these models, effective solution algorithms are developed based on systematic analyses of the synchronism of those functional coefficients, referring to the *VCII* method developed in this study. Besides, synchronisms of interval uncertainties in both constraints and the objective function are evaluated through the proposition of *conditional interval function*s. Sensitivities of *constraint violation likelihood*s, which may vary with constraints due to nonuniformity of unit profits or penalties, are analyzed for enabling reasonable determination of *constraint violation likelihood*s and further optimality of the whole system. Multilevel factorial analysis is integrated with *VCII* to assess the effects of independence of constraint analysis and objective-function analysis. *SILP* can be coupled with techniques such as linearization, artificial intelligence, probability theory, fuzzy set theory, discreteness analysis, fractional optimization, multilevel programming, time series analysis, global sensitivity analysis, and mechanism design. This may be helpful for providing reliable decision support for real-world management problems in which a variety of coupled complexities exist. These improvements or extensions would maximize potential significances of the developed *SILP* method. Currently, synchronic probabilistic linear programming is being studied. Efforts will be made on other problems in future.

8 Conclusions

In this study, an *SILP* method was proposed for optimization of *RESM* problems under synchronic interval uncertainties. Origination, characteristics, influences and quantitative analysis of interval uncertainties in *LP* problems were reviewed. A definition of *interval sets* was developed for reflecting interval uncertainties. *ILP* models were constructed through coupling *interval sets* and *LP* models. Origination, influences and characterization of synchronic interval uncertainties in the programming objective were analyzed. Based on the definition of *interval function*s, an *SILP* model was proposed to formulate *SILP* problems. Analysis of geometric properties of feasible regions of *SILP* models led to proposition of *constraint violation likelihood*s (*CVL*). Constraints of *SILP* models were transformed to deterministic linear inequalities or equalities under given maximal values of *CVL*s (i.e. $CVL_{i\max}$). The *integrally optimal solution* was proposed to quantify the overall optimality of an *SILP* model based on multidimensional integral. Equivalence with existing related definitions, further transformation of *SILP* models, and simplification of models under particular cases were discussed. Through combining these efforts, a *VCII* solution method was exploited for solving *SILP* models. Procedures of *VCII* were specified, and were demonstrated through a simple *RESM* problem. *SILP* was compared with selected existing methods for revealing its effectiveness. Potential improvements and extensions of *SILP* were also assessed.

This study fills the gap of few studies on optimization of *RESM* systems under synchronic interval uncertainties.

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Interval sets defined in this study are capable of addressing interval uncertainties of component properties in linear programming systems. *Interval function*s enable incorporating synchronic effects of interval uncertainties in the programming objective into the optimization process. It is revealed that geometric features of feasible regions under interval uncertainties are independent with distributions of coefficients. Accordingly, *CVL* is proposed based on the geometric analysis. It facilitates quantifying violation of constraints. Analysis of the tradeoff between *constraint violation likelihood*s and system optimality is enabled in the developed *VCII* algorithm. Significant potential negative influences of synchronic interval uncertainties, which may lead to ineffective allocation of resources, a decrease of system optimality, violation of constraints and unexpectedly high costs of recourse measures in real-world problems, are disclosed based on numerical examples and are mitigated in *VCII*. The proposition of multidimensional interval integration in *VCII* achieves the overall optimality of *SILP* problems under synchronic interval uncertainties. The provision of overall optimal solutions associated with a variety of *constraint violation likelihood*s facilitates risk analysis and management of *SILP* problems.

Furthermore, the *SILP* model and the *VCII* method exploit a framework of addressing synchronisms of uncertainties in programming problems. As for other types of uncertainties, e.g. randomness or fuzziness, the synchronic effect also exists in real-world management problems. It is a particular case of interactions or dependence of these uncertainties. The representative feature is that this interaction is easily identified. Depending on the framework of this study, synchronisms of randomness or fuzziness in real-world programming problems can be refined as functional forms, and be incorporated into the optimization process. Based on constraint violation analysis, multidimensional integration of synchronic probabilistic or possibilistic distributions is helpful for providing decision makers with overall optimal management schemes. A series of disasters due to neglecting synchronisms of intervalness, randomness, fuzziness or other uncertainties in decision support processes can be mitigated, which is the most significant contribution of this study.

The proposed *SILP* method in this study is the first attempt to integrate *interval function*s, geometric analysis, and interval integral together for optimization of *SILP* problems. It is deficient in some aspects. For instance, *SILP* is not the desired decision support tool when uncertainties do not exist or when distributional information is known. *SILP* cannot reflect synchronisms of the interval, random or fuzzy coefficients in constraints and the objective function. A quantitative method is lacked to avoid incorrectness in determining acceptable *constraint violation likelihood*s of interval coefficient constraints. Nonlinearities of *interval function*s may challenge transformation of the multidimensional integral of the objective function under synchronic interval uncertainties. Neglecting potential dependency among Interval uncertainties in both constraints and the objective function may affect obtained solutions and their feasibilities. *SILP* in ineffective at addressing the diversity of uncertainties, a multiplicity of programming objectives, discreteness of component properties, dynamics of system features, the nonlinearity of component interrelationships and interactions of them. Efforts will be made to mitigate these challenges, which will start from the proposition of synchronic probabilistic linear programming methods.

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Appendix

Lemma 1 *Solutions under the conservative boundary* are *absolutely feasible, and that out of the optimistic boundary are infeasible.*

Proof Let $\boldsymbol{R}^- = \{\boldsymbol{X} \mid \boldsymbol{A}^+\boldsymbol{X} \le b^- \text{ and } \boldsymbol{X} \ge 0\}$, $\boldsymbol{R}^+ = \{\boldsymbol{X} \mid \boldsymbol{A}^-\boldsymbol{X} \le b^+ \text{ and } \boldsymbol{X} \ge 0\}$, and $\boldsymbol{R}(b, \boldsymbol{A}) = \{\boldsymbol{X} \mid \boldsymbol{AX} \le b; \boldsymbol{A}^- \le \boldsymbol{A} \le \boldsymbol{A}^+; \boldsymbol{X} \ge 0; b^- \le b \le b^+; (b, \boldsymbol{A}) \ne (b^-, \boldsymbol{A}^+) \text{ or } (b^+, \boldsymbol{A}^-); (b^+ - b^-) + \sum_{j=1,2,\ldots,n}(a_j^+ - a_j^-) > 0\}$. The lemma is equivalent with $\boldsymbol{R}^- \subset \boldsymbol{R}(b, \boldsymbol{A}) \subset \boldsymbol{R}^+$. Let $\boldsymbol{X} = (x_1, x_2, \ldots, x_n)$ be any element in $\boldsymbol{R}^-$, and $\boldsymbol{A}$ be a real vector satisfying $\boldsymbol{A}^- \le \boldsymbol{A} \le \boldsymbol{A}^+$. From definitions of $\boldsymbol{R}^-$, we have $\boldsymbol{A}^+\boldsymbol{X} \le b^-$, $\boldsymbol{X} \ge 0$, and $\boldsymbol{AX} \le \boldsymbol{A}^+\boldsymbol{X}$. Namely, $\boldsymbol{AX} \le \boldsymbol{A}^+\boldsymbol{X} \le b^- \le b$ where $b^- \le b \le b^+$. Thus, $\boldsymbol{X} \in \boldsymbol{R}(b, \boldsymbol{A})$, i.e. $\boldsymbol{R}^- \subseteq \boldsymbol{R}(b, \boldsymbol{A})$. Since $(b, \boldsymbol{A}) \ne (b^-, \boldsymbol{A}^+)$, $\boldsymbol{R}^-$ absolutely belongs to $\boldsymbol{R}(b, \boldsymbol{A})$. Let $\boldsymbol{X}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ be any vector in $\boldsymbol{R}(b, \boldsymbol{A})$ and satisfy $\boldsymbol{X}^* \notin \boldsymbol{R}^-$. Similarly, we have $\boldsymbol{X}^* \in \boldsymbol{R}^+$, i.e. $\boldsymbol{R}(b, \boldsymbol{A}) \subseteq \boldsymbol{R}^+$. Accordingly, $\boldsymbol{R}(b, \boldsymbol{A})$ absolutely belongs to $\boldsymbol{R}^+$ because $(b, \boldsymbol{A}) \ne (b^+, \boldsymbol{A}^-)$. □

Proposition 1 *Let* $\boldsymbol{X}$ *be any solution in* $\boldsymbol{R}_s$ *where* $\boldsymbol{R}_s = \{\boldsymbol{X} \mid \boldsymbol{X} \ge 0; \boldsymbol{A}^+\boldsymbol{X} > b^-; \boldsymbol{A}^-\boldsymbol{X} \le b^+; (b^+ - b^-) + \sum_{j=1,2,\ldots,n}(a_j^+ - a_j^-) > 0\}$. *We have* $d_{XT} + d_{XL} > 0$.

Proof For any $\boldsymbol{X} \in \boldsymbol{R}_s$, we have $\boldsymbol{X} \ge 0$, $\boldsymbol{A}^+\boldsymbol{X} > b^-$, $\boldsymbol{A}^-\boldsymbol{X} \le b^+$, and $(b^+ - b^-) + \sum_{j=1,2,\ldots,n}(a_j^+ - a_j^-) > 0$. Since $\boldsymbol{A}^+\boldsymbol{X} - b^- > 0$, $b^+ - \boldsymbol{A}^-\boldsymbol{X} \ge 0$, $(\boldsymbol{A}^+)(\boldsymbol{A}^+)^{\mathrm{T}} > 0$ and $(\boldsymbol{A}^-)(\boldsymbol{A}^-)^{\mathrm{T}} > 0$, we have $d_{XT} \ge 0$ and $d_{XL} \ge 0$. Accordingly, we have $d_{XT} + d_{XL} \ge 0$. If $d_{XT} = 0$ and $d_{XL} = 0$, we have $(\boldsymbol{A}^+\boldsymbol{X} - b^-)/((\boldsymbol{A}^+)(\boldsymbol{A}^+)^{\mathrm{T}}) = 0$ and $(b^+ - \boldsymbol{A}^-\boldsymbol{X})\,/\,((\boldsymbol{A}^-)(\boldsymbol{A}^-)^{\mathrm{T}}) = 0$, i.e. $\boldsymbol{A}^+\boldsymbol{X} = b^-$ and $\boldsymbol{A}^-\boldsymbol{X} = b^+$. Equivalently, $\sum_{j=1,2,\ldots,n}(a_j^+ \cdot x_j) = b^-$, $\sum_{j=1,2,\ldots,n}(a_j^- \cdot x_j) = b^+$, and

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$\sum_{j=1,2,\ldots,n}(a_j^+ - a_j^-)\cdot x_j = b^- - b^+$. Since $\sum_{j=1,2,\ldots,n}(a_j^+ - a_j^-)\cdot x_j \geq 0$ and $b^- - b^+ \leq 0$, equality $\sum_{j=1,2,\ldots,n}(a_j^+ - a_j^-)\cdot x_j = b^- - b^+$ holds for any solution of non-negative decision variables x_j ($j = 1, 2, \ldots, n$) if and only if $b^+ = b^-$ and $a_j^+ = a_j^-$ for any $j \in \{1, 2, \ldots, n\}$. As a result, $(b^+ - b^-) + \sum_{j=1,2,\ldots,n}(a_j^+ - a_j^-) = 0$, which contradicts with the given condition $(b^+ - b^-) + \sum_{j=1,2,\ldots,n}(a_j^+ - a_j^-) > 0$. Thus, it does not hold that $d_{XT} = d_{XL} = 0$. □

Theorem 1 *For any $i \in \{1, 2, \ldots, t\}$, the ith constraint in SILP model* (1) *is equivalent to inequality* (2).

Proof From formulations of CVL_i, we have inequality $CVL_i \leq CVL_{imax}$ is equivalent with $[(A_i^+X - b_i^-)/((A_i^+)(A_i^+)^T)]/\{[(A_i^+X - b_i^-)/((A_i^+)(A_i^+)^T)] + [(b_i^+ - A_i^-X)/((A_i^-)(A_i^-)^T)]\} \leq CVL_{imax}$. Since $X \in R_s$, we have $A^+X > b^-$, $A^-X \leq b^+$, and $(b^+ - b^-) + \sum_{j=1,2,\ldots,n}(a_j^+ - a_j^-) > 0$. Therefore, $[(A_i^+)(A_i^+)^T(b_i^+ - A_i^-X) + (A_i^-)(A_i^-)^T(A_i^+X - b_i^-)] > 0$. Besides, it holds for any $i \in \{1, 2, \ldots, t\}$ that $(A_i^-)(A_i^-)^T \geq 0$ and $(A_i^+)(A_i^+)^T \geq 0$. Thus, we have inequality $[(A_i^+X - b_i^-)/((A_i^+)(A_i^+)^T)]/\{[(A_i^+X - b_i^-)/((A_i^+)(A_i^+)^T)] + [(b_i^+ - A_i^-X)/((A_i^-)(A_i^-)^T)]\} \leq CVL_{imax}$ is equivalent to $[(A_i^+X - b_i^-) / ((A_i^+)(A_i^+)^T)] \leq (CVL_{imax})\{[(A_i^+X - b_i^-) / ((A_i^+)(A_i^+)^T)] + [(b_i^+ - A_i^-X) / ((A_i^-)(A_i^-)^T)]\}$, $(1 - CVL_{imax})(A_i^+X - b_i^-)((A_i^-)(A_i^-)^T) \leq (CVL_{imax})(b_i^+ - A_i^-X)((A_i^+)(A_i^+)^T)$, and then $[(1 - CVL_{imax})(A_i^-)(A_i^-)^TA_i^+ + (CVL_{imax})(A_i^+)(A_i^+)^TA_i^-]X \leq (CVL_{imax})(A_i^+)(A_i^+)^Tb_i^+ + (1 - CVL_{imax})(A_i^-)(A_i^-)^Tb_i^-$. □

Proposition 2 *For any $i \in \{1, 2, \ldots, t\}$, assume $X \in R_s$ and CVL_{imax1} and CVL_{imax2} are any two values of CVL_{imax}. If $CVL_{imax1} < CVL_{imax2}$ and $A_i(CVL_{imax1})X \leq b_i(CVL_{imax1})$, then $A_i(CVL_{imax2})X \leq b_i(CVL_{imax2})$.*

Proof Let X be any vector in $\{X \mid X \in R_s; A_i(CVL_{imax1})X \leq b_i(CVL_{imax1})\}$. Then X satisfies $X \in R_s$ and $A_i(CVL_{imax1})X \leq b_i(CVL_{imax1})$. From Theorem 1, we have inequality $A_i(CVL_{imax1})X \leq b_i(CVL_{imax1})$ is equivalent with $[(A_i^+X - b_i^-)/((A_i^+)(A_i^+)^T)]/\{[(A_i^+X - b_i^-)/((A_i^+)(A_i^+)^T)] + [(b_i^+ - A_i^-X)/((A_i^-)(A_i^-)^T)]\} \leq CVL_{imax1}$. Since $CVL_{imax1} < CVL_{imax2}$, so $[(A_i^+X - b_i^-)/((A_i^+)(A_i^+)^T)]/\{[(A_i^+X - b_i^-)/((A_i^+)(A_i^+)^T)] + [(b_i^+ - A_i^-X)/((A_i^-)(A_i^-)^T)]\} < CVL_{imax2}$. Namely, X belongs to $\{X \mid X \in R_s$ and $A_i(CVL_{imax2})X \leq b_i(CVL_{imax2})\}$. No element in $\{X \mid X \in R_s$ and $A_i(CVL_{imax1})X \leq b_i(CVL_{imax1})\}$ can satisfy $A_i(CVL_{imax2})X = b_i(CVL_{imax2})$, because $CVL_{imax1} < CVL_{imax2}$. Therefore, $\{X \mid X \in R_s$ and $A_i(CVL_{imax1})X \leq b_i(CVL_{imax1})\}$ absolutely belongs to $\{X \mid X \in R_s$ and $A_i(CVL_{imax2})X \leq b_i(CVL_{imax2})\}$. □

Remark 1 For any $CVL_{imax} \in [0, 1]$ and any $i \in \{1, 2, \ldots, t\}$, $L_{ij}(CVL_{imax}) \in [a_{ij}^-, a_{ij}^+]$ and $R_i(CVL_{imax}) \in [b_i^-, b_i^+]$ where $L_{ij}(CVL_{imax}) = [(1 - CVL_{imax})\cdot a_{ij}^+\cdot(A_i^-)(A_i^-)^T + (CVL_{imax})\cdot a_{ij}^-\cdot(A_i^+)(A_i^+)^T]/[(1 - CVL_{imax})\cdot(A_i^-)(A_i^-)^T + (CVL_{imax})\cdot(A_i^+)(A_i^+)^T]$ and $R_i(CVL_{imax}) = [(CVL_{imax})\cdot b_i^+\cdot(A_i^+)(A_i^+)^T + (1 - CVL_{imax})\cdot b_i^-\cdot(A_i^-)(A_i^-)^T]/[(1 - CVL_{imax})\cdot(A_i^-)(A_i^-)^T + (CVL_{imax})\cdot(A_i^+)(A_i^+)^T]$.

Proof Based on formulations of $L_{ij}(CVL_{imax})$ and $R_i(CVL_{imax})$, it is equivalent to prove $[(1 - CVL_{imax})\cdot a_{ij}^+\cdot(A_i^-)(A_i^-)^T + (CVL_{imax})\cdot a_{ij}^-\cdot(A_i^+)(A_i^+)^T] \geq [(1 - CVL_{imax})\cdot(A_i^-)(A_i^-)^T + (CVL_{imax})\cdot(A_i^+)(A_i^+)^T]\cdot a_{ij}^-$, $[(1 - CVL_{imax})\cdot a_{ij}^+\cdot(A_i^-)(A_i^-)^T + (CVL_{imax})\cdot a_{ij}^-\cdot(A_i^+)(A_i^+)^T] \leq [(1 - CVL_{imax})\cdot(A_i^-)(A_i^-)^T + (CVL_{imax})\cdot(A_i^+)(A_i^+)^T]\cdot a_{ij}^+$, $[(CVL_{imax})\cdot b_i^+\cdot(A_i^+)(A_i^+)^T + (1 - CVL_{imax})\cdot b_i^-\cdot(A_i^-)(A_i^-)^T] \geq [(1 - CVL_{imax})\cdot(A_i^-)(A_i^-)^T + (CVL_{imax})\cdot(A_i^+)(A_i^+)^T]\cdot b_i^-$, and $[(CVL_{imax})\cdot b_i^+\cdot(A_i^+)(A_i^+)^T + (1 - CVL_{imax})\cdot b_i^-\cdot(A_i^-)(A_i^-)^T] \leq [(1 - CVL_{imax})\cdot(A_i^-)(A_i^-)^T + (CVL_{imax})\cdot(A_i^+)(A_i^+)^T]\cdot b_i^+$. Namely, $[(1 - CVL_{imax})\cdot a_{ij}^+\cdot(A_i^-)(A_i^-)^T] \geq [(1 - CVL_{imax})\cdot(A_i^-)(A_i^-)^T]\cdot a_{ij}^-$, $[(CVL_{imax})\cdot a_{ij}^-\cdot(A_i^+)(A_i^+)^T] \leq [(CVL_{imax})\cdot(A_i^+)(A_i^+)^T]\cdot a_{ij}^+$, $[(CVL_{imax})\cdot b_i^+\cdot(A_i^+)(A_i^+)^T] \geq [(CVL_{imax})\cdot(A_i^+)(A_i^+)^T]\cdot b_i^-$, and $[(1 - CVL_{imax})\cdot b_i^-\cdot(A_i^-)(A_i^-)^T] \leq [(1 - CVL_{imax})\cdot(A_i^-)(A_i^-)^T]\cdot b_i^+$. These inequalities hold because, for any $i \in \{1, 2, \ldots, t\}$ and any $j \in \{1, 2, \ldots, n\}$, we have $1 - CVL_{imax} \geq 0$, $CVL_{imax} \geq 0$, $(A_i^-)(A_i^-)^T \geq 0$, $(A_i^+)(A_i^+)^T \geq 0$, $a_{ij}^+ \geq a_{ij}^-$ and $b_i^+ \geq b_i^-$. □

Theorem 2 *$DCVL(CVL_{imax}) < DCVL(CVL_{imax})$ if $CVL_{imax1} < CVL_{imax2}$ where CVL_{imax1} and CVL_{imax2} are two levels of CVL_{imax}.*

Proof Due to the formulation of $DCVL(CVL_{imax})$, it is sufficient to prove that both $F_{ij}(L_{ij}(CVL_{imax}))$ and $G_i(R_i(CVL_{imax}))$ are monotonically decreasing with CVL_{imax} for any $i \in \{1, 2, \ldots, t\}$ and $j \in \{1, 2, \ldots, n\}$. It is equivalent to prove that a) $L_{ij}(CVL_{imax1}) \geq L_{ij}(CVL_{imax2})$ and b) $R_i(CVL_{imax1}) \leq R_i(CVL_{imax2})$, since both $F_{ij}(\cdot)$ and $G_i(\cdot)$ are monotonically increasing functions.

(a). Since $CVL_{imax1} \leq CVL_{imax2}$ and $a_{ij}^- \leq a_{ij}^+$, we have $(CVL_{imax1} - CVL_{imax2})\cdot(a_{ij}^- - a_{ij}^+) \geq 0$. Because $(A_i^+)(A_i^+)^T\cdot(A_i^-)(A_i^-)^T > 0$, we have $(CVL_{imax1})\cdot(1 - CVL_{imax2})\cdot a_{ij}^- + (1 - CVL_{imax1})\cdot(CVL_{imax2})\cdot a_{ij}^+ \geq (CVL_{imax2})\cdot(1 - CVL_{imax1})\cdot a_{ij}^- + (1 - CVL_{imax2})\cdot(CVL_{imax1})\cdot a_{ij}^+$. Equivalently, we have $(1 - CVL_{imax1})\cdot(1 - CVL_{imax2})\cdot a_{ij}^+\cdot(A_i^-)(A_i^-)^T\cdot(A_i^-)(A_i^-)^T + [(CVL_{imax1})\cdot(1 - CVL_{imax2})\cdot a_{ij}^- + (1 - CVL_{imax1})\cdot(CVL_{imax2})\cdot a_{ij}^+]\cdot(A_i^+)(A_i^+)^T\cdot(A_i^-)(A_i^-)^T + (CVL_{imax1})\cdot(CVL_{imax2})\cdot a_{ij}^-\cdot(A_i^+)(A_i^+)^T\cdot(A_i^+)(A_i^+)^T \geq (1 - CVL_{imax1})\cdot(1 - CVL_{imax2})\cdot a_{ij}^+\cdot(A_i^-)(A_i^-)^T\cdot(A_i^-)(A_i^-)^T + [(1 - CVL_{imax1})\cdot(CVL_{imax2})\cdot a_{ij}^- + (CVL_{imax1})\cdot(1 - CVL_{imax2})\cdot a_{ij}^+]\cdot(A_i^+)(A_i^+)^T\cdot(A_i^-)(A_i^-)^T + (CVL_{imax1})\cdot(CVL_{imax2})\cdot a_{ij}^-\cdot(A_i^+)(A_i^+)^T\cdot(A_i^+)(A_i^+)^T$, and furthermore $(1 - CVL_{imax1})\cdot a_{ij}^+\cdot(A_i^-)(A_i^-)^T\cdot(1 - CVL_{imax2})\cdot(A_i^-)(A_i^-)^T + (CVL_{imax1})\cdot a_{ij}^-\cdot(A_i^+)(A_i^+)^T\cdot(1 - CVL_{imax2})\cdot(A_i^-)(A_i^-)^T + (1 - CVL_{imax1})\cdot a_{ij}^+\cdot(A_i^-)(A_i^-)^T\cdot(CVL_{imax2})\cdot(A_i^+)(A_i^+)^T + (CVL_{imax1})\cdot a_{ij}^-\cdot(A_i^+)(A_i^+)^T\cdot(CVL_{imax2})\cdot(A_i^+)(A_i^+)^T \geq (1 - CVL_{imax2})\cdot a_{ij}^+\cdot(A_i^-)(A_i^-)^T\cdot(1 - CVL_{imax1})\cdot(A_i^-)(A_i^-)^T + (CVL_{imax2})\cdot a_{ij}^-\cdot(A_i^+)(A_i^+)^T\cdot(1 - CVL_{imax1})\cdot$

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$(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (1 - CVL_{imax2})\cdot a_{ij}^+\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T\cdot(CVL_{imax1})\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T + (CVL_{imax2})\cdot a_{ij}^-\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T\cdot(CVL_{imax1})\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T$. Since $[(1 - CVL_{imax})\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax})\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T] \geq 0$ for any $CVL_{imax} \in [0, 1]$ and any $i \in \{1, 2, \ldots, t\}$, we have $[(1 - CVL_{imax1})\cdot a_{ij}^+\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax1})\cdot a_{ij}^-\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T]\cdot[(1 - CVL_{imax2})\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax2})\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T] \geq [(1 - CVL_{imax2})\cdot a_{ij}^+\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax2})\cdot a_{ij}^-\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T]\cdot[(1 - CVL_{imax1})\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax1})\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T]$. Thus, $[(1 - CVL_{imax1})\cdot a_{ij}^+\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax1})\cdot a_{ij}^-\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T]/[(1 - CVL_{imax1})\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax1})\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T] \geq [(1 - CVL_{imax2})\cdot a_{ij}^+\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax2})\cdot a_{ij}^-\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T]/[(1 - CVL_{imax2})\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax2})\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T]$. It is equivalent to $L_{ij}(CVL_{imax1}) \geq L_{ij}(CVL_{imax2})$ from the formulation of $L_{ij}(CVL_{imax})$. □

(b). From the formulation of $R_i(CVL_{imax})$, we are going to prove $[(CVL_{imax1})\cdot b_i^+\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T + (1 - CVL_{imax1})\cdot b_i^-\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T]/[(1 - CVL_{imax1})\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax1})\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T] \leq [(CVL_{imax2})\cdot b_i^+\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T + (1 - CVL_{imax2})\cdot b_i^-\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T]/[(1 - CVL_{imax2})\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax2})\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T]$. That is, $[(CVL_{imax1})\cdot b_i^+\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T + (1 - CVL_{imax1})\cdot b_i^-\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T]\cdot[(1 - CVL_{imax2})\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax2})\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T] \leq [(CVL_{imax2})\cdot b_i^+\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T + (1 - CVL_{imax2})\cdot b_i^-\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T]\cdot[(1 - CVL_{imax1})\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax1})\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T]$, or $(CVL_{imax1})\cdot b_i^+\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T\cdot(1 - CVL_{imax2})\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (1 - CVL_{imax1})\cdot b_i^-\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T\cdot(1 - CVL_{imax2})\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax1})\cdot b_i^+\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T\cdot(CVL_{imax2})\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T + (1 - CVL_{imax1})\cdot b_i^-\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T\cdot(CVL_{imax2})\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T \leq [(CVL_{imax2})\cdot b_i^+\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T\cdot(1 - CVL_{imax1})\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (1 - CVL_{imax2})\cdot b_i^-\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T\cdot(1 - CVL_{imax1})\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax2})\cdot b_i^+\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T\cdot(CVL_{imax1})\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T + (1 - CVL_{imax2})\cdot b_i^-\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T\cdot(CVL_{imax1})\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T$. It is equivalent to $[(CVL_{imax1})\cdot(1 - CVL_{imax2})\cdot b_i^+ + (1 - CVL_{imax1})\cdot b_i^-\cdot(CVL_{imax2})]\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (1 - CVL_{imax1})\cdot(1 - CVL_{imax2})\cdot b_i^-\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax1})\cdot(CVL_{imax2})\cdot b_i^+\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T \leq [(1 - CVL_{imax1})\cdot(CVL_{imax2})\cdot b_i^+ + (CVL_{imax1})\cdot(1 - CVL_{imax2})\cdot b_i^-]\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (1 - CVL_{imax1})\cdot(1 - CVL_{imax2})\cdot b_i^-\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T + (CVL_{imax1})\cdot(CVL_{imax2})\cdot b_i^+\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T\cdot(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T$. Since $CVL_{imax1} \leq CVL_{imax2}$ and $b_i^- \leq b_i^+$, we have $[(CVL_{imax1})\cdot(1 - CVL_{imax2})\cdot b_i^+ + (1 - CVL_{imax1})\cdot b_i^-\cdot(CVL_{imax2})] - [(1 - CVL_{imax1})\cdot(CVL_{imax2})\cdot b_i^+ + (CVL_{imax1})\cdot(1 - CVL_{imax2})\cdot b_i^-] = (CVL_{imax1} - CVL_{imax2})\cdot(b_i^+ - b_i^-)$ and $(CVL_{imax1} - CVL_{imax2})\cdot(b_i^+ - b_i^-) \leq 0$. At the meantime, $(\mathbf{A}_i^+)(\mathbf{A}_i^+)^T\cdot(\mathbf{A}_i^-)(\mathbf{A}_i^-)^T \geq 0$. Thus, we have $R_i(CVL_{imax1}) \leq R_i(CVL_{imax2})$. □

Theorem 3 *If the necessarily optimal solution* exists for *SILP-2 model* (3)*, the integrally optimal solution* equals to the *necessarily optimal solution*.

Proof Let $\mathbf{X}_{opt} = (x_{1opt}, x_{2opt}, \ldots, x_{nopt})$ be the *integrally optimal solution* of model (3). From Definition 6, we have $\int\ldots\int\left\{\sum_{j=1}^{n}\left[\left[g_j(d_1, d_2, \ldots, d_r) + h_j\right]x_j\right]\right\}d(d_1)\ldots d(d_r)$ is maximized when $\mathbf{X} = \mathbf{X}_{opt}$. Namely, there does not exist another feasible solution $\mathbf{X}' = (x_1', x_2', \ldots, x_n')$ such that $\int\ldots\int\left\{\sum_{j=1}^{n}\left[\left[g_j(d_1, d_2, \ldots, d_r) + h_j\right]x_j'\right]\right\}d(d_1)\ldots d(d_r) > \int\ldots\int\left\{\sum_{j=1}^{n}\left[\left[g_j(d_1, d_2, \ldots, d_r) + h_j\right]x_{jopt}\right]\right\}d(d_1)\ldots d(d_r)$. Assume $\mathbf{X}_{opt}$ is not the *necessarily optimal solution* that exists for *SILP-2* model (3). From the definition of *necessarily optimal solution*, there must exist another vector of feasible solutions, assumed as $\mathbf{X}'' = (x_1'', x_2'', \ldots, x_n'')$, such that $\sum_{j=1}^{n}\left[\left[g_j(d_1, d_2, \ldots, d_r) + h_j\right]x_j''\right] \geq \sum_{j=1}^{n}\left[\left[g_j(d_1, d_2, \ldots, d_r) + h_j\right]x_{jopt}\right]$ for all combinations of $d_k \in [d_k^-, d_k^+]$ $(k = 1, 2, \ldots, r)$ and $\sum_{j=1}^{n}\left[\left[g_j(d_1, d_2, \ldots, d_r) + h_j\right]x_j''\right] > \sum_{j=1}^{n}\left[\left[g_j(d_1, d_2, \ldots, d_r) + h_j\right]x_{jopt}\right]$ for at least one combination of $d_k \in [d_k^-, d_k^+]$ $(k = 1, 2, \ldots, r)$. Thus, we have $\int\ldots\int\left\{\sum_{j=1}^{n}\left[\left[g_j(d_1, d_2, \ldots, d_r) + h_j\right]x_j''\right]\right\}d(d_1)\ldots d(d_r) > \int\ldots\int\left\{\sum_{j=1}^{n}\left[\left[g_j(d_1, d_2, \ldots, d_r) + h_j\right]x_{jopt}\right]\right\}d(d_1)\ldots d(d_r)$. This is contradictory to the maximization of $\int\ldots\int\left\{\sum_{j=1}^{n}\left[\left[g_j(d_1, d_2, \ldots, d_r) + h_j\right]x_{jopt}\right]\right\}d(d_1)\ldots d(d_r)$. Therefore, the *integrally optimal solution* is also the *necessarily optimal solution*. □

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